Black Sun Rising

Investigations into the Hawking Effect

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Unified Field Theory

Tim Joseph
In the beginning there was Aristotle,
And objects at rest tended to remain at rest,
And objects in motion tended to come to rest,
An soon everything was at rest,
And God saw that it was boring.

Then God created Newton,
And objects at rest tended to remain at rest,
But objects in motion tended to remain in motion,
And energy ws conserved and momentum was conserved and matter was conserved,
And God saw that it was conservative.

Then God created Einstein,
And everything was relative,
And fast things became short,
And straight things became curved,
And the universe was filled with inertial frames,
And God saw that it was relatively general, but some of it was especially relative.

The God created Bohr,
And there was the principle,
And the principle was quantum,
And all things were quantized,
But some things were still relative,
And God saw that it was confusing.

Then God was going to create Fergeson,
And Fergeson would have unified,
And he would have fielded a theory,
All all would have been one,
But it was the seventh day,
And God rested,
And objects at rest tend to remain at rest.

Contents

Abstract

We introduce quantum field theory on curved spacetimes and consider the fields of Klein and Gordon, Dirac and Maxwell upon black hole spacetimes. Assymptotic field solutions are found for all of these fields and particle emission from the vacuum derived on their basis. A new physical effect of the emission of scalars from a cluster of black holes is found and discussed. The emission spectra are compared and analysed with regard to their physical content, observability and the information paradox. No prior knowledge of quantum field theory or general relativity is necessary.

Preface

"Essentia non sunt multiplicanda praeter necessitatem."

- William of Ockham.

Before Hawking's prediction in 1975 that black holes radiate energy [17], a black hole was the ultimate rubbish bin. Everything thrown in would never emerge again and communication from the inside was utterly impossible. The combination of general relativity and quantum field theory brought about the prediction that black holes do radiate their energy content away in the form of particles. This prediction has caused the rising of the black sun on the heavens of eternity.

The result caused a great stir in theoretical physics because it was the first concrete prediction of a limiting case of the elusive theory of everything. Nevertheless much is to be learned about the radiation of black holes. The Hawking effect, strictly refers to the emission of massless scalars from a Schwarzschild black hole, this is the simplest situation imaginable. One would need to investigate the emission of other particles from a more general black hole to observe some features that may lie hidden in the approximations. This is the purpose of the present investigation.

Since we are bound by constraints of time and space for this report, certain assumptions must be made of the readership of this article. Knowledge of basic quantum mechanics, classical mechanics (particularly the Lagrangian and Hamiltonian formulations) and the variational calculus is essential. A number of branches of mathematics such as real and complex analysis, partial and ordinary differential equations will be used frequently and knowledge of them is necessary. No knowledge of quantum field theory, relativistic quantum mechanics, or general relativity will be assumed whereas it would be very helpful if the reader had a basic grounding in these theories.

The necessary elements of quantum field theory and general relativity will be reviewed in the first chapter. This is a brief introduction or reminder so that the formalism of the following chapters is set in context. The concept of a black hole will be introduced in some detail. Chapter one also introduces the general basis for quantum field theory on curved spacetimes to the extent that we will make use of it.

Chapter two will discuss the scalar field on a general black hole spacetime. Chapter three will do the same for the Dirac and Maxwell fields. Chapter four treats the scalar field on a spacetime of a cluster of N black holes. Chapter five considers the gravitational and self interaction of quantum fields on a general black hole spacetime. Chapter six compares the various results of the preceeding chapters mathematically and graphically. Some important features of the emission spectra are discussed, including a new, unpublished effect discovered as part of this project. The famous information paradox will then be discussed and some motivations and directions for future research suggested.

Appendix A

Introduction

> *"The miracle of the appropriateness of language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."*
>
> - Eugene Wigner.

Relativistic quantum mechanics is a very successful theory but it is difficult to perform calculations involving many particles. This limitation gave the initial impetus to generate quantum field theory (QFT). Treating many particles as one entity, the field, simplifies the mathematics considerably. One may quantise the theory by introducing creation and annihilation operators for field states in the same manner as for the non-relativistic quantum mechanical harmonic oscillator. This approach is particularly useful when dealing with the electromagnetic field. Photons emerge very naturally from the quantisation of the Maxwell field, whereas they would be difficult to treat normally since particle number is not conserved for bosons [4].

QFT is a particularly successful theory and builds the theoretical basis of modern particle physics. Quantum electrodynamics (QED), an element of QFT, treats the interaction of the electromagnetic field with the fields for the atomic particles, the Dirac field. Nevertheless, QFT integrally assumes the background spacetime of special relativity and is thus not immediately compatible with general relativity (GR). The 'marriage' of QFT and GR is a major goal of theoretical physics and is thought to take the form of a theory of everything in which all phenomena could be explained. This unification of the interactions or forces has proven elusive until now. However, as long as the curvature of space is small compared to the wavelength of the field under investigation, we may treat the gravitational field classically, that is via general relativity. The only basic change is then to render the field equation into a covariant form in a general curved spacetime and deriving a new quantum field theory in curved spacetime (QFTCS) from it. This work is still progressing but much has been done by a variety of people. Because curved spaces are much more complicated than flat ones, one may expect a myriad of new physical effects from the theory. Most new things are very formal or esoteric and it is difficult to interpret the intricate formalism physically but one effect stands out: the Hawking effect. This effect predicts that black holes are not black; that they emit particles continuously due to a property of the solutions of the field equations of QFTCS.

Elements of QFT and GR are reviewed below as a brief introduction or reminder for the reader. They are not pedagogical treatments but will be sufficient if one simply wishes to read this exposition of the Hawking effect. Only those parts of QFT and GR, which are necessary for this paper, will be reviewed. A short discussion of the formalism of QFTCS is also given in the third section of this chapter. This is, by no means, complete but serves as a sufficiently detailed introduction to the material treated later.

A.1 General Relativity and Black Holes

"Black holes are the bungholes of space."

- Butthead.

A space is principally characterised by the function which calculates the distance between two points, the metric or line-element . When this is not equivalent to Pythagoras' theorem, we have a non-Euclidean geometry in which the last of Euclid's axioms (two parallel lines never cross) does not necessarily hold. This theory was first proposed by Riemann [28] and so a large class of these spaces are known as Riemannian spaces and thus we have a Riemannian geometry in them. According to Riemann, the metric is the main characteristic of a space; in addition to its dimensionality and boundary properties. In general, a line element dl^2 may be written

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{A.1}$$

where $g_{\mu\nu}$ is known as the metric tensor. In (A.1), we have made use of the Einstein summation convention which dictates to perform a sum across all those indices appearing once as a subscript and once as a superscript in one term of an expression; thus there is a double sum implied in (A.1). For our purposes, we will be working in a four-dimensional spacetime and hence $\mu, \nu \in \{1, 2, 3, 4\}$. If we are working in spherical polar coordinates, we may identify the coordinates $x^\mu \in \{t, r, \theta, \phi\}$.The special case of the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.2}$$

is the Minkowski metric tensor [24] and specifies special relativity, whereas a general metric tensor, in which all the elements may be functions of all x^μ, leaves us in the domain of general relativity proper.

If one knows $g_{\mu\nu}$ and the matter content of spacetime, then one may use general relativity to predict events in the same manner as one would use Newton's laws in ordinary classical mechanics. We shall not delve into general relativity but take it and its solutions for granted. For a full treatment of the theory and its solutions see [25]. Indeed, we do not need to make use of the Einstein field equations at all, we will be concerned only with specific metrics.

There are several methods of deriving the expression for the metric tensor of a single massive object with a non-zero electric charge Q and angular momentum per unit mass $a \equiv J/M$. We shall not derive it, but state it

$$g_{\mu\nu} = \begin{pmatrix} \frac{B}{\Sigma} - 1 & 0 & 0 & -\frac{aB\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{aB\sin^2\theta}{\Sigma} & 0 & 0 & \left[r^2 + a^2 + \frac{a^2 B\sin^2\theta}{\Sigma}\right]\sin^2\theta \end{pmatrix} \tag{A.3}$$

where

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 \tag{A.4}$$

$$\Sigma \equiv r^2 + a^2\cos^2\theta \tag{A.5}$$

$$B \equiv 2Mr - Q^2 \tag{A.6}$$

This is known as the Kerr-Newman solution [25]. Two other solutions may be derived from it: The Kerr solution by setting $Q = 0$ and the Schwarzschild solution by, in addition, setting $a = 0$. All three solutions represent black holes. There exists a famous theorem that black holes have no hair [7] , which means that the three parameters M, a, and Q completely describe the most general black hole possible. The black hole, in this manner, represents the most 'perfect' object in the universe. There is no complication by a very large number of constituent particles and complex internal dynamics. The above metric for the Kerr-Newman solution to Einstein's field equations is thus the most general metric for a black hole and will be a central piece of the following investigations. First, we need to determine a few properties of the metric.

A black hole is black since nothing may escape its gravitational pull; we shall see later that this is not completely true, but in the classical limit it is. The distance away from the centre of the hole (the origin $r = 0$ in (A.3)) where light may just escape from the hole is known as the event horizon. This distance r_E may be determined by solving the equation $\Delta = 0$. Since this is a quadratic, there are two solutions, the larger is r_E and the smaller is known as the Cauchy horizon r_C. The Cauchy horizon is important in the theory of black holes, but will not be important here; it suffices to say that $r_C = 0$ only for the Schwarzschild solution. For the event horizon, we obtain

$$r_E = M + \sqrt{M^2 - a^2 - Q^2} \tag{A.7}$$

We will also need to determine the inverse of the metric tensor, which is defined by $g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda_\nu$, where δ^λ_ν is the Kroenecker delta symbol. So the inverse of the metric tensor is numerically equivalent to the inverse of the matrix representation of the tensor. Equally well, we may denote $g \equiv \det g_{\mu\nu}$. Thus we find

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{a^2 \sin^2\theta - \left(r^2 + a^2\right)^2}{\Delta\Sigma} & 0 & 0 & -\dfrac{aB}{\Delta\Sigma} \\ 0 & \dfrac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\dfrac{aB}{\Delta\Sigma} & 0 & 0 & \dfrac{\Sigma - B}{\Delta\Sigma\sin^2\theta} \end{pmatrix} \tag{A.8}$$

$$g = -\Sigma^2 \sin^2\theta \tag{A.9}$$

In his exposition of non-Euclidean geometry, Riemann invented a tensor that completely specifies the curvature of a space. This Riemann tensor $R^\alpha_{\beta\gamma\delta}$ may be calculated from the metric alone via the connection coefficients or Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g_{\alpha\mu} \left(\frac{\partial g_{\mu\beta}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu} \right) \tag{A.10}$$

$$R^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}. \tag{A.11}$$

When $R^{\alpha}_{\beta\gamma\delta} = 0$, the space is called flat and is similar to the space of special relativity in which two parallel lines will never cross. For black holes, we can determine that the Riemann tensor satiesfies

$$\lim_{r\to\infty} R^{\alpha}_{\beta\gamma\delta} = 0 \tag{A.12}$$

Thus the black hole spacetime is *assymptotically flat* and special relativity is valid an infinite distance away from the black hole. We shall see that this is an important property later on.

To aid visualisation of black holes, Roger Penrose developed the method of conformal diagrams in which the entire spacetime may be drawn in a finite picture. He achieves this by transforming the metric tensor through a conformal transformation defined by

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \tag{A.13}$$

where the function $\Omega^2(x)$ is known as the conformal factor. For a Schwarzschild black hole, we may obtain a simple picture of the spacetime shown in figure 1.1.

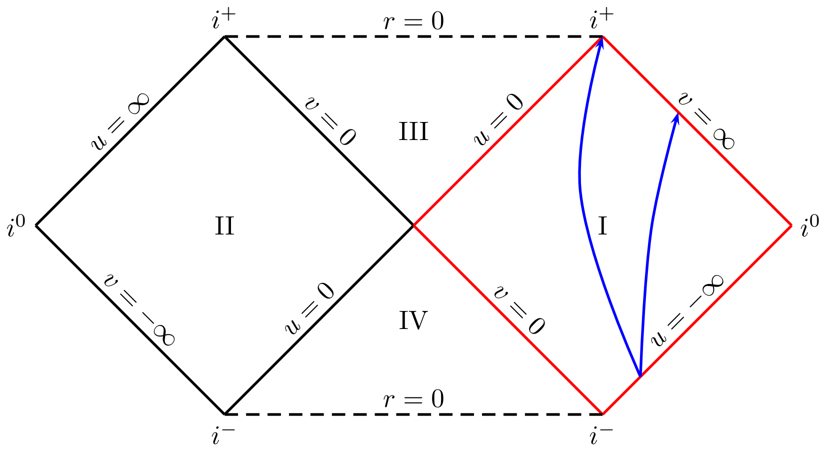

Figure 1.1: The conformal representation of the Schwarzschild spacetime. The dashed lines represent the singularity and the solid lines the future and past event horizons. Physical particles are limited to the red diamond and the two blue curves show possible trajectories of particles.

Black holes, as we know them, were first predicted by Karl Schwarzschild who discovered that the Schwarzschild metric was an exact solution to Einstein's equations for a spherically symmetric massive object. Through the mathematical discovery of the event horizon from this solution, the idea of black holes was re-invented from Descartes idea that a sufficiently massive object should be able to bend lightrays so that they must return to it after a finite distance. For many years, black holes were an entirely theoretical construct kept alive by the success of general relativity in other areas. Today there is much evidence that they actually exist at the centre of galaxies, that they might be quasars and responsible for gamma-ray bursters. The reader is directed to the literature for this information [36] [27] [23].

This completes the general relativity we must know to proceed within the realms of this project. All we will need to do later is to generalise the d'Alembertian and Dirac operators to curved spacetimes and in particular to the Kerr-Newman spacetime characterised by the metric (A.3). Using further analysis of this metric and making use of some general relativity proper, one may derive many properties of black holes including many interesting features of particle trajectories, these are outside the scope of this project but are treated extensively in [8].

A.2 QFT in Minkowski Space

"Science is built up of facts, as a house is built of stones;
but an accumulation of facts is no more a science than
a heap of stones is a house."

- Henri Poincaré.

To construct the theory, we write down a Lagrangian $L(q,\dot{q})$ for some generalised coordinate q, where $\dot{q} = dq/dt$. From the Lagrangian, we construct the action A by

$$A = \int_{t_1}^{t_2} L(q,\dot{q})dt. \tag{A.14}$$

The action principle then requires that A be stationary, i.e. that $\delta A = 0$. This requirement gives us the Euler-Lagrange equations of motion [14]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{A.15}$$

Often, we may express the Lagrangian in terms of a Lagrangian density $\mathcal{L}$ such that

$$L \equiv \int_{-\infty}^{\infty} d^3x \mathcal{L}\left(\phi, \frac{\partial \Phi}{\partial x^\mu}\right) \tag{A.16}$$

where Φ is the field and the x^μ are the field coordinates. Then (A.15) becomes

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \left(\partial \Phi / \partial x^\mu\right)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0. \tag{A.17}$$

In QFT, we work mainly in terms of $\mathcal{L}$ and so (A.15) is the equation for the field that we must consider relevant.

In principle, the aim is to solve the Euler-Lagrange equations (A.17) for Φ. Once Φ has been found, one knows everything about the system since Φ is the equivalent of the wavefunction of quantum mechanics. In general, of course, this can not be done and different methods need to be found to extract the required information. Many such methods have been developed and the reader is advised to seek out specialised textbooks for them, such as [5]. To really be a QFT, the theory must also be 'quantised' so that an interpretation of the solutions to (A.17) in terms of particles becomes possible. This is usually done by finding a particular solution and then writing the general solution as a sum over these particular solutions modified by unkown functions which may be interpreted (for the correct choice of particular solution) as creation and annihilation operators in a very similar way as one does for the simple quantum mechanical harmonic oscillator. Then one may determine the energy spectrum and other properties of the field without ever finding a completely general solution to (A.17) in closed form.

To illustrate this, let us choose a specific form for $\mathcal{L}$

$$\mathcal{L}\left(\Phi, \frac{\partial \Phi}{\partial x^\mu}\right) = \frac{1}{2}\left(\frac{\partial \Phi}{\partial x^\mu}\frac{\partial \Phi}{\partial x_\mu} - m^2\Phi^2\right), \tag{A.18}$$

the Euler-Lagrange equations (A.17) give us the Klein-Gordon equation

$$\left(\Box^2 + m^2\right)\Phi = 0 \tag{A.19}$$

where $\Box^2$ is the d'Alembertian operator, which is given by

$$\Box^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{A.20}$$

in Minkowski space, where ∇^2 is the Laplacian.

For a physical theory, we must not choose a 'free-field' Lagrangian as, for example, in (A.18) but must add an interaction term. Only through interactions may a theory be verified and become useful, so we express

$$\mathcal{L}_{tot} = \mathcal{L}_{free} + \mathcal{L}_{self} + \mathcal{L}_{int} \tag{A.21}$$

where $\mathcal{L}_{int}$ is the interaction of the field with other fields or external influencing mechnisms and $\mathcal{L}_{self}$ represents the self-interaction of the field. $\mathcal{L}_{int}$ is very general and changes between different situations but $\mathcal{L}_{self}$ is most often expressed as

$$\mathcal{L}_{self} = -\frac{1}{4}\lambda\Phi^4 \tag{A.22}$$

which is imaginatively known as the $\lambda\Phi^4$ theory. The solutions to the field equations become more difficult now, of course, but one may determine some real physical effects. Using different expressions for $\mathcal{L}_{int}$, one may treat the weak, strong and electromagnetic interactions quantum mechanically [33].

A.3 QFT in Curved Spacetime

"We hope to explain the entire universe in a single, simple formula that you can wear on your T-shirt."

- Leon Lederman.

We wish to combine QFT and GR in such a way that the particle fields are quantised and the gravitational field is taken into account simply through the incorporation of a different, coordinate-dependant metric tensor. In general, this is a complicated procedure because it is not immediately obvious how the interpretation generalises to a curved space. This is primarily because one needs to define a direction along which one can usefully define a positive frequency [38]. This direction ς is usually chosen such that the field is an eigenfunction of the Lie derivative along that direction [32]

$$\widetilde{L_\varsigma}\Phi = -i\omega\Phi \tag{A.23}$$

One may show that a so-called Killing vector is always a candidate for ς and it is in general assumed that one must have a Killing vector in order to have a solution to (A.23). These Killing vectors are solutions to the differential equation [31]

$$\widetilde{\nabla}_\alpha\varsigma_\beta + \widetilde{\nabla}_\beta\varsigma_\alpha = 0 \tag{A.24}$$

where $\widetilde{\nabla}_\alpha$ is the covariant derivative. In general, (A.24) has no solutions but in the case of asympotically flat spacetimes, a Killing vector may always be found and the interpretation of the quantum field theory becomes easier. For black holes, there are two Killing vectors

$$\varsigma = \frac{\partial}{\partial t}, \ \frac{\partial}{\partial \phi} \tag{A.25}$$

The first vector in (A.25) is a good choice since it is identical to the choice of Killing vector made in ordinary QFT. Thus the interpretation is identical when a solution of the field equations is found. The question of interpretation is an intricate one for spacetimes that do not have a Killing vector $\varsigma = \frac{\partial}{\partial t}$ and becomes generally impossible for spacetimes that are not assymptotically flat. Since black holes satisfy both of these requirements, we may proceed to formulate QFTCS.

First, one must define a curved space Lagrangian density $\mathcal{L}$ and then find the field equation via the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \left(\partial \Phi / \partial x^\mu\right)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0. \tag{A.26}$$

Equally, we may generalise the scalar product between two fields

$$(\Phi_1, \Phi_2) = -i \int_\Sigma \Phi_1 \overline{\partial_\mu} \Phi_2^* \left(-g_\Sigma\right)^{\frac{1}{2}} n^\mu d\Sigma \tag{A.27}$$

where n^μ is a future-directed unit vector normal to the spacelike hypersurface Σ, $d\Sigma$ is the volume element in Σ and

$$\Phi_1 \overline{\partial_\mu} \Phi_2^* = \Phi_1 \partial_\mu \Phi_2^* - \left[\partial_\mu \Phi_1\right] \Phi_2^* \tag{A.28}$$

is the Wronskian with respect to x^μ. It can be shown that (A.27) is independant of the choice of Σ as long as it is a Cauchy surface in a globally hyperbolic spacetime [16]. For the case of black hole spacetimes, a suitable choice for Σ is a sphere centered on the black hole. The simplest possible choice for n^μ is then $n^\mu = (1, 0, 0, 0)$.

It is possible to find a complete set of orthonormal mode functions $\{u_i\}$ with respect to the scalar product (A.27) which satisfy (A.26). The orthonormality properties can be expressed as

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0, \tag{A.29}$$

where * indicates complex conjugation. Since the set $\{u_i\}$ is complete, we may expand the field in the u_i

$$\Phi = \sum_i \left(a_i u_i + a_i^\dagger u_i^*\right). \tag{A.30}$$

where we impose the covariant quantisation via the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad \text{etc.} \tag{A.31}$$

and † indicates Hermitian conjugation. We can also find another complete set $\{\bar{u}_i\}$ which obeys the relations (A.29) - (A.31). With an analogy to the harmonic oscillator, it can be seen that the operators a_i and $a_i^\dagger$ (and of course $\bar{a}_i$ and $\bar{a}_i^\dagger$) are the annihilation and creation operators respectively. Since the sets $\{u_i\}$ and $\{\bar{u}_i\}$ are different complete sets, they both define different vacuum states (i.e. the states which can not be further annihilated.)

$$a_i \left|0\right\rangle = 0, \quad \bar{a}_j \left|\bar{0}\right\rangle = 0, \qquad \forall\, i, j\ \in I^+ \tag{A.32}$$

The action of the creation operator $a_i^\dagger$ upon the vacuum state $\left|0\right\rangle$ would then result in the one-particle state in the mode u_i. Similarly because of the completeness property, we may expand one set in terms of the other

$$u_i = \sum_j \left(\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*\right) \qquad \bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right). \tag{A.33}$$

Such a transformation is known as a Bogolubov transformation [2], where the matrices α_{ij} and β_{ij} are the Bogolubov coefficients. By projecting out, we find that

$$\alpha_{ij} = (\bar{u}_i, u_j) \qquad \beta_{ij} = -\left(\bar{u}_i, u_j^*\right) \tag{A.34}$$

and the expectation value

$$\left\langle\bar{0}\right| N_i \left|\bar{0}\right\rangle = \left\langle\bar{0}\right| a_i^\dagger a_i \left|\bar{0}\right\rangle = \sum_j \left|\beta_{ji}\right|^2 \tag{A.35}$$

where $N_i = a_i^\dagger a_i$ is number operator with respect to the set $\{u_i\}$. The physical interpretation of (A.35) is that the vacuum state $\left|\bar{0}\right\rangle$ of the set $\{\overline{u}_i\}$ contains $\sum_j \left|\beta_{ji}\right|^2$ particles in the mode u_i. In order to investigate particle creation from the vacuum state, the principal aim must be to evaluate the Bogolubov coefficient of the second kind β_{ij}. Indeed (A.35) is the emission spectrum of the spacetime from one set of modes into another. If we choose the two sets of solutions such that they have added physical significance, we may be able to attach more meaning to the emission spectrum between these two families of modes. For example, if we choose

$$u_i = u_i^{(in)} = \lim_{t\to-\infty} \Phi \qquad \overline{u}_i = \overline{u}_i^{(out)} = \lim_{t\to\infty} \Phi \tag{A.36}$$

we may be able to interpret (A.35) as genuine particle creation by the spacetime curvature. This occurs because we have defined both sets of modes in terms of the same general solution in different limits of time. The "in" modes $u_i^{(in)}$ are those in the assymptotic past and the "out" modes $\overline{u}_i^{(out)}$ are those in the assymptotic future. In Minkowski space, these limits would be identical and thus no particle creation would occur, however in curved spacetime this may not be true. It may very well be that the "in" and "out" modes have a different form and can thus be non-trivially expanded in terms of another giving nonzero Bogolubov coefficients. An observer far away from the black hole would see the "out" states, which would contain a different number of particles than the "in" modes. This is particularly strange for the vacuum state which would not be empty with respect to the "in" vacuum. During the transition through the black hole, the vacuum state has gained particles. This is a particle creation by the spacetime curvature itself and this is what we shall seek to find later. In relativistic quantum mechanics, one requires positive frequency solutions to remain with positive frequency at all times and vice versa. This is in an effort to keep an electron from changing into a positron randomly. Here, we may get a mixed state of positive and negative frequency modes and thus we have created particles.

For definiteness, let us take the scalar field and start with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(-g)^{\frac{1}{2}}\left[g^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Phi}{\partial x^\nu} - m^2\Phi^2\right] \tag{A.37}$$

Via (A.26), we obtain the curved spacetime equivalent of the Klein-Gordon equation (A.19)

$$\left(\Box^2 + m^2\right)\Phi = 0 \tag{A.38}$$

For a general metric, the d'Alembertian becomes

$$\Box^2\Phi = (-g)^{-\frac{1}{2}}\,\partial_\mu\left[(-g)^{\frac{1}{2}}\,g^{\mu\nu}\partial_\nu\Phi\right] \tag{A.39}$$

where g is given by (A.9) and $\partial_\mu \equiv \partial/\partial x^\mu$. For the scalar case, we would need to solve (A.38) for the "in" and the "out" modes and then find the Bogolubov coefficients of the second kind to get the particle creation spectrum.

The creation of particles from the vacuum may be pictured by an electron-positron pair creation near the event horizon of a black hole. The positron becomes trapped and the electron escapes, thus it would seem as if an electron has been created. See figure 1.2 for an illustration and the Feynman diagram for such a process.

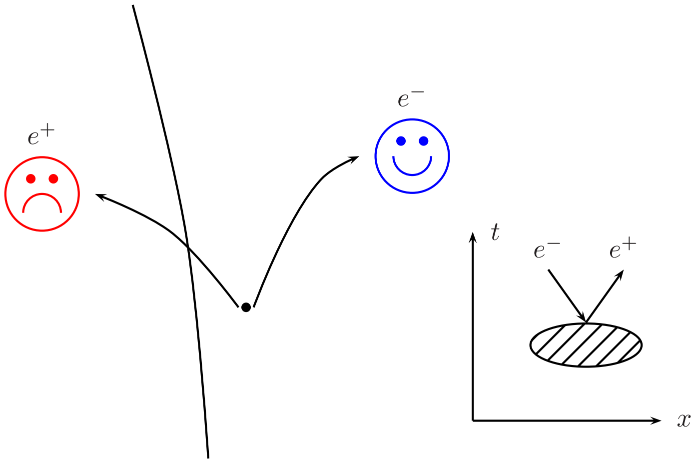

Figure 1.2: An electron-positron pair is produced from the vacuum near the event horizon of a black hole. The negative energy positron falls into the black hole, while the positive energy electron escapes. This is one possible interpretation of the particle emission properties of the black hole spacetimes. The appropriate Feynman diagram is drawn on the lower right corner [4].

Appendix B

The Klein-Gordon Scalar Field

"I may not understand it, but it sure looks important to me."

- Ian Stewart.

The scalar field was originally proposed independantly by five different groups in 1926 as the theory for an spinless, neutral and massive particle [15] [12] [19] [22] and [10]. It was also proposed by Schrödinger when he rendered his famous equation compatible with relativity. So it is really to be thought of as the relativistic theory of the same particle that one considers in quantum theory when one solves the Schrödinger equation. Through a twist of history, only the names of Klein and Gordon have become attached to the equation. The term scalar is used here to indicate that the solution is one function and not a spinor or tensor as in the case of the Dirac or Maxwell fields. This automatically means that the particle must be spinless. One may treat a charged, spinless particle via a complex superposition of two Klein-Gordon fields, but we shall not treat such an extension here. For a lengthy discussion of the field in Minkowski spacetime, see [33].

B.1 The Scalar Field in Schwarzschild Spacetime

"I know that hardly any physicists believe that the gravitational forces can play any part in the constitution of matter. The physicist always argues that the forces are too small. This reminds me of a joke. An unmarried woman had a child and the family was greatly humiliated. So the midwife tried to console the mother by saying: 'Don't worry so much, it's a very small child!'"

- Albert Einstein.

The Lagrangian for the Klein-Gordon field is given by (A.37) and the metric for the Schwarzschild geometry is obtained by setting $a = Q = 0$ in the expressions (A.3) - (A.9) for the Kerr-Newman geometry . We then get the Klein-Gordon equation from (A.38)

$$\left[\frac{r^3}{2M-r}\partial_t^2 + \partial_r\left[r\left(r-2M\right)\partial_r\right]\right]\Phi + \left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\phi^2 + m^2r^2\right]\Phi = 0 \quad \text{(B.1)}$$

This is a separable equation and so we obtain

$$\Phi\left(t,r,\theta,\phi\right) = e^{-i\omega t}R(r)Y_{ln}\left(\theta,\phi\right) \quad \text{(B.2)}$$

where we have chosen the time dependancy such that (A.23) is satisfied for the Killing vector $\varsigma = \partial/\partial t$. The functions $Y_{ln}\left(\theta,\phi\right)$ are the spherical harmonics and the radial function satisfies

$$\partial_r^2 R + \frac{2\left(r-M\right)}{r\left(r-2M\right)}\partial_r R + \left[m^2r^2 - l\left(l+1\right) + \frac{\omega^2r^2}{r-2M}\right]\frac{1}{r\left(r-2M\right)}R = 0 \quad \text{(B.3)}$$

We may put $R(r) = r^{-1}F(r)$ and transform to a new coordinate $\rho(r) = r + 2M\ln\left|\frac{2M-r}{2M}\right|$ to obtain the equation

$$\partial_\rho^2 F(\rho) + \left[\frac{m^2}{r} - \frac{l(l+1)}{r^3} + \frac{\omega^2}{r-2M} + \frac{2M}{r^4}\right](r-2M)F(\rho) = 0 \quad \text{(B.4)}$$

The solutions we want are far from the black hole and so we only consider the solutions as the radial coordinate $r \to \infty$.

$$\partial_\rho^2 F(\rho) + \left[m^2 + \omega^2\right] F(\rho) = 0 \tag{B.5}$$

The general solution of (B.5) is

$$F(\rho) = Ae^{-i\alpha\rho} + Be^{i\alpha\rho} \tag{B.6}$$

where $\alpha^2 = m^2 + \omega^2$. For the Schwarzschild spacetime, there are two null coordinates, which we may generalise to

$$u = t - \left[\left(\frac{m}{\omega}\right)^2 + 1\right]^{\frac{1}{2}} \rho \tag{B.7}$$

$$v = t + \left[\left(\frac{m}{\omega}\right)^2 + 1\right]^{\frac{1}{2}} \rho \tag{B.8}$$

We may write the field solution in terms of them and thus we have two complete sets of solutions, one valid in the assymptotic past and one in the assymptotic future. Thus we have found the "in" and the "out" modes of the scalar field for large times and large distances away from the black hole.

$$u_i = u_{\omega ln}^{(in)} = r^{-1} e^{-i\omega v} Y_{ln}(\theta, \phi) \tag{B.9}$$

$$\overline{u}_i = u_{\omega ln}^{(out)} = r^{-1} e^{-i\omega u} Y_{ln}(\theta, \phi) \tag{B.10}$$

Bearing in mind that the scalar product (A.27) is independant of the choice of hypersurface, it can be shown that both sets of solutions are indeed complete sets and may be suitably normalised according to (A.29). We note that the solution in terms of spherical harmonics has forced both l and n to be integers such that $-l \leq n \leq l$. There is no restriction upon ω or the range of values of l. Thus the field may have a continuous range of frequencies and may exist in all angular momentum states $0 \leq l < \infty$.

B.2 Computing the Bogolubov Coefficients

"Now you may ask, 'What is mathematics doing in a physics lecture?' We have several possible excuses: first, of course, mathematics is an important tool, but that would only excuse us for giving the formula in two minutes. On the other hand, in theoretical physics we discover that all our laws can be written in mathematical form; and that this has a certain simplicity and beauty about it. So, ultimately, in order to understand nature it may be necessary to have a deeper understanding of mathematical relationships. But the real reason is that the subject is enjoyable, and although we humans cut nature up in different ways, and we have different courses in different departments, such compartmentalization is really artificial, and we should take our intellectual pleasures where we find them."

- Richard P. Feynman.

Having computed the assymptotic modes, we must evaluate the Bogolubov coefficients of the second kind. This is a rather complex integral and may be simplified if we can expand the null coordinates in terms of each other and then perform an integral over the complex plane. Consider a collapsing shell of matter, then outside the shell, we would have the Schwarzschild metric

$$dl^2 = dt'^2 - dr^2 - r^2 d\Omega^2 \tag{B.11}$$

and inside the shell, the normal Minkowski metric in spherical polar coordinates

$$dl^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{B.12}$$

However the spacetime must have a continuous metric across any boundary to satisfy a basic philosophical need of a physical spacetime. We may thus easily derive, that

$$\frac{R-2M}{R}\left(\frac{dt}{dt'}\right)^2 + \left[1 - \frac{R}{R-2M}\right]\left(\frac{dR}{dt'}\right)^2 = 1 \tag{B.13}$$

where $r = R(t)$ must be satisfied. Figure 2.1 shows such a shell at two times and the null coordinates passing through. By the continuity condition, we must thus connect the curved null coordinate v to the flat one v' which must be connected to the flat null coordinate u' and that to its curved equivalent u. If we then follow the chain of connections in reverse, we may find $u = u(v)$. As it turns out, because of the forms of the flat null coordinates, all the connections are just linear except the final one in which we need to make use of the metric continuity condition (B.13).

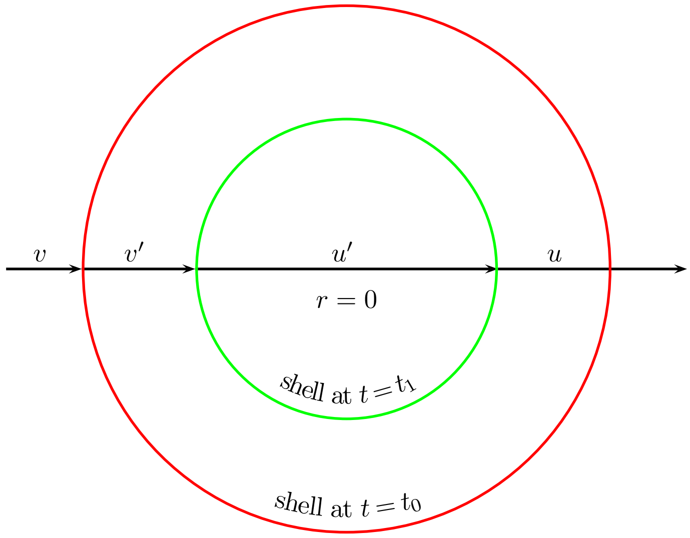

Figure 2.1: The null vectors change when they intersect the collapsing shell. First we have v which changes into v' when it intersects the shell at time t_0. v' changes into u' when it crosses the origin and then into u as it crosses the shell at time t_1 with $t_1 > t_0$, which has collapsed further in the intervening time.

We define $t'_0 \equiv t'_{(R=2M)}$ and so near $t' = t'_0$ we may put $R(t') \sim 2M + a'(t'_0 - t')$. Then the continuity condition (B.13) gives

$$t \sim -2M \ln\left(\frac{t'_0 - t'}{2M}\right) \tag{B.14}$$

which enables us to deduce the form of u

$$u = u(v) = -4M\alpha \ln\left|\frac{v_0 - v}{2MC}\right| \tag{B.15}$$

where C is a constant and

$$\alpha = \frac{1}{2}\left(1 + \left[\left(\frac{m}{\omega}\right)^2 + 1\right]^{\frac{1}{2}}\right) \tag{B.16}$$

Both modes may now be expressed in terms of the null coordinate v and we may write the Bogolubov transformation (A.33). By projecting out the modes in turn, we may easily derive the Bogolubov coefficients

$$\alpha_{\omega\omega' ln} = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_0^\infty e^{-i\omega' \widetilde{v}} \exp\left[4Mi\omega\alpha \ln\left|\frac{\widetilde{v}}{2MC}\right|\right] d\widetilde{v} \tag{B.17}$$

$$\beta_{\omega\omega' ln} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} e^{i\omega v_0} \int_0^\infty e^{i\omega' \widetilde{v}} \exp\left[4Mi\omega\alpha \ln\left|\frac{\widetilde{v}}{2MC}\right|\right] d\widetilde{v} \tag{B.18}$$

where $\widetilde{v} = v_0 - v$. We now regard $\widetilde{v}$ as a complex variable and do the integrals in the complex plane. Using properties of complex integrals, we may determine that we can transfer one of the above integrals into the other by a contour-deformation. For a general function $f(\widetilde{v})$ and constant b [30]

$$\left|\int_{\mathcal{C}} e^{-i\omega' \widetilde{v}} \exp\left(bif\right) d\widetilde{v}\right| = -e^{b}\left|\int_{\mathcal{C}} e^{i\omega' \widetilde{v}} \exp\left(bif\right) d\widetilde{v}\right| \tag{B.19}$$

where $\mathcal{C}$ is the infinite semi-circle in the upper-half plane. Thus

$$|\alpha_{\omega\omega' ln}| = e^{4\pi M\omega\alpha}\,|\beta_{\omega\omega' ln}| \tag{B.20}$$

But the Bogolubov coefficients are normalised via

$$\sum_{\omega'}\left(|\alpha_{\omega\omega' ln}|^2 - |\beta_{\omega\omega' ln}|^2\right) = 1 \tag{B.21}$$

and so

$$N_{\omega ln} = \sum_{\omega'} |\beta_{\omega\omega' ln}|^2 = \left[e^{8\pi M\omega\alpha} - 1\right]^{-1} \tag{B.22}$$

The particle production spectrum (B.22) is the number of particles of mass m created by a Schwarzschild black hole of mass M into a mode of angular frequency ω

$$N_\omega = \left[\exp\left(\frac{\omega}{T}\right) - 1\right]^{-1} \tag{B.23}$$

we obtain a frequency dependant temperature

$$T(\omega) = \frac{1}{4\pi M}\left[\left(\left(\frac{m}{\omega}\right)^2 + 1\right)^{\frac{1}{2}} + 1\right]^{-1} \tag{B.24}$$

We see that if $m = 0$, we have a genuine Planck spectrum with a constant temperature but if the field mass does not vanish, then the spectrum deviates from the Planck one . The celebrated Hawking effect [17] states that a massless scalar field is emitted from a Schwarzschild black hole in an exactly Planck spectrum. When we go back to geometrical units and put in the surface gravity of the black hole $\kappa = (4M)^{-1}$ and consider a massless field, we obtain

$$T = \frac{\kappa}{2\pi} \tag{B.25}$$

which is precisely the Hawking result.

B.3 The Scalar Field in Kerr-Newman Spacetime

"A black hole has no hair."

- John A. Wheeler.

Now we reconsider the scalar field in the general black hole case. In the same manner as above, we consider the free Klein-Gordon field equations and evaluate the d'Alembertian (A.39) for the metric (A.3). When we separate the variables according to

$$\Phi = e^{-i\omega t} e^{in\phi} S_{n\omega}(\theta) R(r) \tag{B.26}$$

We find that the functions $S_{n\omega}(\theta)$ and $R(r)$ satisfy the equations

$$\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) - \frac{n^2}{\sin^2\theta} - a^2\omega^2\sin^2\theta + A\right] S_{n\omega}(\theta) = 0 \tag{B.27}$$

and

$$\partial_r\left(\Delta\partial_r R\right) + \left[\omega^2\left(r^2 + a^2\right)^2 + a^2n^2 + m^2\Sigma\Delta - 2an\omega B - A\Delta\right]\frac{R}{\Delta} = 0 \tag{B.28}$$

If we put $\eta = \cos\theta$ and $\lambda_\omega = A - a^2\omega^2$ we may transform the angular equation (B.27) into the standard for for the spheroidal wavefunctions $S_{n\omega} = S_{n\omega}(a, \cos\theta)$

$$\left[\partial_\eta \left[\left(1-\eta^2\right)\partial_\eta\right] + \lambda_\omega - \frac{n^2}{1-\eta^2} + a^2\eta^2\right] S_{n\omega}(a, \cos\theta) = 0 \tag{B.29}$$

the properties of which may be looked up in [1].

The radial equation will be treated as before. We put $R(r) = r^{-1}F(r)$ and transform to a coordinate $\rho(r)$ which satisfies

$$\frac{d\rho}{dr} = \frac{r^2 + a^2}{\Delta} \tag{B.30}$$

and then obtain the radial equation

$$\begin{aligned} &\partial_\rho^2 F - \left[\frac{2}{r} - \frac{2r\Delta}{\left(r^2+a^2\right)^2}\right]\partial_\rho F \\ &+ \left[\frac{2}{r^2} - \frac{2\Delta}{\left(r^2+a^2\right)^2} + \omega^2 + \frac{a^2n^2 + m^2\Sigma\Delta - 2an\omega B - A\Delta}{\left(r^2+a^2\right)^2}\right] F = 0 \end{aligned} \tag{B.31}$$

which in the limit $r \to \infty$ becomes

$$\partial_\rho^2 F(\rho) + \left[m^2 + \omega^2\right] F(\rho) = 0 \tag{B.32}$$

This is the same assymptotic behaviour in terms of the different coordinate $\rho(r)$ as before in the Schwarzschild case . So we define our null coordinates in an annalogous manner to (B.7) and thus get the "in" and "out" modes

$$u_i = u_{\omega n\lambda}^{(in)} = r^{-1}e^{-i\omega v} S_{n\omega}(a, \cos\theta) \tag{B.33}$$

$$\overline{u}_i = u_{\omega n\lambda}^{(out)} = r^{-1}e^{-i\omega u} S_{n\omega}(a, \cos\theta) \tag{B.34}$$

Similarly to the Schwarzschild case, it may be shown that both solutions independantly form a complete set of solutions according to (A.29). As before, there is no restriction placed upon the frequency of the field ω.

B.4 Evaluating the Emission Spectrum

"For what one can measure exists."

- Max Planck.

Again, we must connect the null coordinates across the boundary of a collapsing shell to find $u \equiv u(v)$. This time, of course, the shell is rotating and charged in addition to being massive. The metric within, therefore, takes the form

$$dl^2 = -\left(1 + \frac{Q^2}{\Sigma}\right) dt^2 + \frac{2Q^2 a \sin\theta}{\Sigma} dt d\phi + \frac{\Sigma}{r^2 + a^2 + Q^2} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 - \frac{Q^2 a^2 \sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2 \quad \text{(B.35)}$$

Since the horizon is located at $r_E = M + \sqrt{M^2 - a^2 - Q^2}$ we may put $R(t') = r_E + A\,(t'_0 - t')$ in addition to setting $\phi\,(t') = B$, where A and B are constants. We can then deduce that

$$\left(\frac{dt}{dt'}\right)^2 \sim \frac{4M^4}{(r_E - M)^2} \left(\frac{1}{t'_0 - t'}\right)^2 \quad \text{(B.36)}$$

which enables us to deduce the form of u analogously to (B.15)

$$u = u(v) = -4M\alpha \ln\left|\frac{v_0 - v}{Cr_E}\right| \quad \text{(B.37)}$$

where C is a constant and

$$\alpha = \frac{1}{2}\left(\left[\left(\frac{m}{\omega}\right)^2 + 1\right]^{\frac{1}{2}} + \frac{M}{r_E - M}\right) \quad \text{(B.38)}$$

The calculations in the complex plane, to find the Bogolubov coefficients, is the same as in equations (B.17) to (B.22) with α given by (B.38). We may therefore deduce the emission spectrum to be identical to (B.22) but with the changed α. The frequency dependant temperature (B.24) is then

$$T\,(\omega) = \frac{1}{4\pi M}\left(\left[\left(\frac{m}{\omega}\right)^2 + 1\right]^{\frac{1}{2}} + \frac{M}{\sqrt{M^2 - a^2 - Q^2}}\right)^{-1} \quad \text{(B.39)}$$

Since the surface gravity of a Kerr-Newmann black hole is given by

$$\kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{2Mr_E - Q^2} \quad \text{(B.40)}$$

one can see that there does not exist a general version of (B.25). If we put $Q = 0$, then we may indeed find that for a massless field

$$T = \frac{\kappa}{2\pi} \tag{B.41}$$

Nevertheless, (B.39) is still a remarkably simple equation for such a complicated object as the Kerr-Newman spacetime.

Appendix C

The Dirac Field in Kerr-Newman Spacetime

"I consider that I understand an equation when I can predict the properties of its solutions, without actually solving it."

- Paul Adrien Maurice Dirac.

The Dirac equation was originally proposed by P.A.M. Dirac, who devised it as the relativistic equation for the electron. While the concept of spin needs to be artificially introduced into non-relativistic quantum mechanics, the Dirac equation has the spin of the electron built in. The equation looks like

$$(i\widetilde{\gamma}^{\mu}\partial_{\mu} - m)\,\Phi = 0 \tag{C.1}$$

where m is the mass of the electron and $\widetilde{\gamma}^{\mu}$ are the Dirac matrices chosen such that they satisfy the anticommutation relation

$$\{\widetilde{\gamma}^{\mu}, \widetilde{\gamma}^{\nu}\} = 2\eta^{\mu\nu} \tag{C.2}$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor given by (A.2). These are four by four constant matrices which satisfy many properties based on (C.2). In fact, many properties of the solution of the Dirac equation may be deduced from the properties of the Dirac matrices which hinge upon their anti-commutation relation. Since each Dirac matrix has 16 components, there is a lot of freedom in choosing the exact values of the matrix components. Each choice is known as a representation of the solution of the Dirac equation - a four component spinor. The most popular such representation, in which most of relativistic quantum mechanics is done, is defined by

$$\widetilde{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\widetilde{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \widetilde{\gamma}^4 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{C.3}$$

C.1 Tetrads and the Dirac Equation

"As God calculates, so the world is made."

- Gottfried Wilhelm Leibniz.

The incorporation of spin into curved spacetimes is most readily done via the Newman-Penrose tetrad formalism. A tetrad is a combination of four vectors **l**, **n**, **m** and $\overline{\mathbf{m}}$, where the overline indicates complex conjugation. We devise a notation for the tetrad $e^i_{(a)}$ in which the index within parenthesis indicates a label for the vector and the free index the components of the vectors, so that

$$e^i_{(1)} = l^i, \qquad e^i_{(2)} = n^i, \qquad e^i_{(3)} = m^i, \qquad e^i_{(4)} = \overline{m}^i \tag{C.4}$$

The tetrad index may the raised and lowered via the metric tensor

$$e_{(a)i} = g_{ik} e^k_{(a)} \tag{C.5}$$

A summation over the tetrad index yields a summation matrix $\widetilde{\eta}_{(a)(b)}$ which is not necessarily numerically equivalent to the Minkowski metric tensor

$$e^i_{(a)} e_{(b)i} = \widetilde{\eta}_{(a)(b)} \tag{C.6}$$

Via the equations (C.5) and (C.6), we may represent the metric tensor as

$$g_{\mu\nu} = e^{(a)}_\mu e^{(b)}_\nu \widetilde{\eta}_{(a)(b)} \tag{C.7}$$

If the matrix $\widetilde{\eta}_{(a)(b)}$ is numerically equivalent to the Minkowski metric , we may verify that the four matrices defined by

$$\gamma^{\mu} = e^{\mu}_{(a)} \widetilde{\gamma}^{(a)} \tag{C.8}$$

satisfy the anti-commuation relation

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \tag{C.9}$$

which is just the curved spacetime generalisation of (C.2). So the matrices γ^{μ} may be regarded as the curved spacetime Dirac matrices . The Dirac equation in curved spacetime takes the form [3]

$$\left[\gamma^{\mu} \left(\partial_{\mu} - \Gamma_{\mu}\right) + m\right] \Phi = 0 \tag{C.10}$$

where γ^{μ} are now defined by (C.8) and Γ_{μ}, the "spin-connection" is defined by

$$\Gamma_{\mu} = \frac{1}{2} \Sigma^{(\alpha)(\beta)} e^{\nu}_{(\alpha)} \left[\frac{\partial}{\partial x^{\mu}} e_{(\beta)\nu}\right] \tag{C.11}$$

where $\Sigma^{(\alpha)(\beta)}$, the generator of the Lorentz group, is defined by

$$\Sigma^{(\alpha)(\beta)} = \frac{1}{4} \left[\gamma^{(\alpha)}, \gamma^{(\beta)}\right] \tag{C.12}$$

Sometimes the spin-connection is also defined equivalently as

$$\Gamma_{\mu} = \frac{1}{4} g_{(\lambda)(\alpha)} \left[e^{(\alpha)}_{\beta} \frac{\partial}{\partial x^{\mu}} e^{\beta}_{(\nu)} - \Gamma^{(\alpha)}_{(\nu)\mu}\right] s^{(\lambda)(\nu)} \tag{C.13}$$

where $s^{(\alpha)(\beta)} = 2\Sigma^{(\alpha)(\beta)}$ and $\Gamma^{(\alpha)}_{(\nu)\mu}$ are the Christoffel symbols from general relativity [37].

There are two main choices for the tetrad: the null tetrad and the Minkowski tetrad . The null tetrad is defined by requiring the four tetrad vectors to be null and normalised. Thus we get a summation matrix (C.6)

$$\widetilde{\eta}_{(a)(b)} = \widetilde{\eta}^{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{C.14}$$

The Minkowski tetrad is defined such that the summation matrix (C.6) is numerically equivalent to the Minkowski metric

$$\eta_{(a)(b)} = \eta^{(a)(b)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.15}$$

This Minkowski tetrad formalism allows us to immediately obtain the curved spacetime Dirac matrices via (C.8).

In curved spacetime, it is desirable to have a Minkowski tetrad because of the easy transformation properties of the Dirac matrices from Minkowski space to curved space. Via knowledge of the spacetime, however it is often easier to find a null tetrad. Thus it is necessary to know if one may proceed from one to the other with relative ease. Let us define a matrix $\alpha^{(a)}_{(b)}$ such that

$$\alpha^{(a)}_{(c)}\alpha^{(b)}_{(d)}\widetilde{\eta}_{(a)(b)} = \eta_{(c)(d)} \tag{C.16}$$

If $\alpha^{(a)}_{(b)}$ is found, we may transform a null tetrad $\widetilde{e}^{i}_{(a)}$ into a Minkowski tetrad $e^{i}_{(a)}$ via

$$e^{i}_{(a)} = \alpha^{(b)}_{(a)}\widetilde{e}^{i}_{(b)} \tag{C.17}$$

and it may be verified that the new tetrad satisfies the relations (C.5) - (C.7) with the Minkowski summation matrix. If we define $A^{(a)(f)} = \alpha^{(a)}_{(c)}\widetilde{\eta}^{(c)(f)}$, we may determine that

$$\underline{A}^2\underline{\widetilde{\eta}} = \underline{\eta} \tag{C.18}$$

Through much algebra, one may determine A^2 and then A by taking the square root of the matrix, to obtain

$$A_{(a)(b)} = \frac{1}{2}\begin{pmatrix} \sqrt{2}i & -\sqrt{2}i & 0 & 0 \\ \sqrt{2}i & \sqrt{2}i & 0 & 0 \\ 0 & 0 & (1-i) & (1+i) \\ 0 & 0 & (1+i) & (1-i) \end{pmatrix} \tag{C.19}$$

and by inverting $\underline{\widetilde{\eta}}$ we find, from the definition of $\underline{A}$

$$\alpha^{(a)}_{(c)} = \frac{1}{2}\begin{pmatrix} -\sqrt{2}i & \sqrt{2}i & 0 & 0 \\ \sqrt{2}i & \sqrt{2}i & 0 & 0 \\ 0 & 0 & -(1+i) & -(1-i) \\ 0 & 0 & -(1-i) & -(1+i) \end{pmatrix} \tag{C.20}$$

Thus, having found a null tetrad we multiply it by $\alpha^{(a)}_{(c)}$ given by (C.20) to get the Minkowski tetrad and get the Dirac matrices via (C.8) from the Minkowski space representation.

C.2 The Dirac Gamma Matrices and the Spin-connection

> *"**The Dirac Method of capturing a lion in the middle of the Sahara Desert:** "We observe that wild lions are, ipso facto, not observable in the Sahara Desert. Consequently, if there are any lions in the Sahara, they are tame. The capture of a tame lion may be left as an exercise for the reader.'"*
>
> \- unknown.

The reason why it is easier to find a null tetrad is because in addition to the vectors obeying many requirements, we may use the null geodesics to construct them. The null geodesics are the lines along which photons travel in a given spacetime and may be found from the metric [25]. In Kerr-Newman geometry , the null geodesics may be found to satisfy [8]

$$\frac{dt}{d\tau} = \frac{r^2+a^2}{\Delta}E, \qquad \frac{dr}{d\tau} = \pm E, \qquad \frac{d\theta}{d\tau} = 0, \qquad \frac{d\phi}{d\tau} = \frac{a}{\Delta}E \tag{C.21}$$

Thus we may choose a set of four vectors based on (C.21) and the requirement that they be null

$$\mathbf{l}\cdot\mathbf{l} = \mathbf{n}\cdot\mathbf{n} = \mathbf{m}\cdot\mathbf{m} = \overline{\mathbf{m}}\cdot\overline{\mathbf{m}} = 0 \tag{C.22}$$

orthogonal

$$\mathbf{l}\cdot\mathbf{m} = \mathbf{l}\cdot\overline{\mathbf{m}} = \mathbf{n}\cdot\mathbf{m} = \mathbf{n}\cdot\overline{\mathbf{m}} = \mathbf{0} \tag{C.23}$$

and normalised

$$\mathbf{l}\cdot\mathbf{n} = -\mathbf{m}\cdot\overline{\mathbf{m}} = \mathbf{1} \tag{C.24}$$

to have a summation matrix given by (C.14). So we obtain the null tetrad for the Kerr-Newman black hole

$$\widetilde{e}^{i}_{(a)} = \begin{pmatrix} \frac{r^2+a^2}{\Delta} & 1 & 0 & \frac{a}{\Delta} \\ \frac{r^2+a^2}{2\Sigma} & -\frac{\Delta}{2\Sigma} & 0 & \frac{a}{2\Sigma} \\ \frac{ia\sin\theta}{\sqrt{2}\rho} & 0 & \frac{1}{\sqrt{2}\rho} & \frac{i\csc\theta}{\sqrt{2}\rho} \\ -\frac{ia\sin\theta}{\sqrt{2}\overline{\rho}} & 0 & \frac{1}{\sqrt{2}\overline{\rho}} & \frac{i\csc\theta}{\sqrt{2}\overline{\rho}} \end{pmatrix} \tag{C.25}$$

where $\rho = r + ia\cos\theta$. We transform the null tetrad into a Minkowski tetrad via (C.17)

$$e^i_{(a)} = \frac{i}{2\sqrt{2\Sigma}} \begin{pmatrix} r^2+a^2 & -\Delta & 0 & a \\ \left(r^2+a^2\right)\left(\dfrac{2\Sigma}{\Delta}+1\right) & 2\Sigma-\Delta & 0 & a\left(\dfrac{2\Sigma}{\Delta}+1\right) \\ 2iap'\sin\theta & 0 & 2ip & -2p\csc\theta \\ 2iap\sin\theta & 0 & 2iap' & 2p'\csc\theta \end{pmatrix} \tag{C.26}$$

where $p = a\cos\theta + r$ and $p' = a\cos\theta - r$. From the Minkowski space Dirac matrices (C.3), we can now determine the Dirac matrices in the Kerr-Newman geometry via (C.8). We obtain

$$\gamma^1 = \frac{i}{2\sqrt{2\Sigma}} \begin{pmatrix} a & 0 & -\Delta & r^2+a^2 \\ a\left(\dfrac{2\Sigma}{\Delta}+1\right) & 0 & 2\Sigma-\Delta & \left(r^2+a^2\right)\left(\dfrac{2\Sigma}{\Delta}+1\right) \\ -2p\csc\theta & 2ip & 0 & 2iap'\sin\theta \\ 2p'\csc\theta & 2ip' & 0 & 2iap\sin\theta \end{pmatrix} \tag{C.27}$$

$$\gamma^2 = \frac{i}{2\sqrt{2\Sigma}} \begin{pmatrix} a & 0 & -\Delta & -\left(r^2+a^2\right) \\ a\left(\dfrac{2\Sigma}{\Delta}+1\right) & 0 & 2\Sigma-\Delta & -\left(r^2+a^2\right)\left(\dfrac{2\Sigma}{\Delta}+1\right) \\ -2p\csc\theta & -2ip & 0 & -2iap'\sin\theta \\ 2p'\csc\theta & -2ip' & 0 & -2iap\sin\theta \end{pmatrix} \tag{C.28}$$

$$\gamma^3 = \frac{i}{2\sqrt{2\Sigma}} \begin{pmatrix} 0 & -a & r^2+a^2 & \Delta \\ 0 & -a\left(\dfrac{2\Sigma}{\Delta}+1\right) & \left(r^2+a^2\right)\left(\dfrac{2\Sigma}{\Delta}+1\right) & \Delta-2\Sigma \\ 2ip & 2p\csc\theta & 2iap'\sin\theta & 0 \\ 2ip' & -2p'\csc\theta & 2iap\sin\theta & 0 \end{pmatrix} \tag{C.29}$$

$$\gamma^4 = \frac{-i}{2\sqrt{2\Sigma}} \begin{pmatrix} r^2+a^2 & -\Delta & 0 & -a \\ \left(r^2+a^2\right)\left(\dfrac{2\Sigma}{\Delta}+1\right) & 2\Sigma-\Delta & 0 & -a\left(\dfrac{2\Sigma}{\Delta}+1\right) \\ 2iap'\sin\theta & 0 & -2ip & 2p\csc\theta \\ 2iap\sin\theta & 0 & -2ip' & -2p'\csc\theta \end{pmatrix} \tag{C.30}$$

This is, of course, one particular representation of the gamma matrices which corresponds to the usual representation (C.3) in Minkowski space. It is possible that there exists a more convinient representation in the Kerr-Newman space. The Minkowski tetrad with lowered indices can then be found via (C.5)

$$e_{(a)i} = \frac{i}{2\sqrt{2}\Sigma}\begin{pmatrix} -\Delta & -\Sigma & 0 & a\Delta\sin^2\theta \\ -\Delta-2\Sigma & \left(\dfrac{2\Sigma}{\Delta}-1\right)\Sigma & 0 & (2\Sigma+\Delta)\,a\sin^2\theta \\ 2a\sin\theta D & 0 & 2ip\Sigma & \begin{array}{r} -2a^2\sin^3\theta A\,(p+ip') \\ -2p\sin\theta\left(r^2+a^2\right) \end{array} \\ 2a\sin\theta D' & 0 & 2ip'\Sigma & \begin{array}{r} 2a^2\sin^3\theta A\,(p'-ip) \\ +2p'\sin\theta\left(r^2+a^2\right) \end{array} \end{pmatrix} \tag{C.31}$$

where $D = Ap + i\,(A-1)\,p'$ and $D' = i(A-1)p - Ap'$ and we have put $A = \left(2Mr - Q^2\right)\Sigma^{-1}$. Throughout this we have assumed that the vector potential is zero. A non-zero vector potential may always be introduced by the transformation

$$\partial_\mu \rightarrow \partial_\mu - iA_\mu \tag{C.32}$$

where A_μ the vector potential is a solution of the massless Klein-Gordon equation. If we are to ignore the vector potential, we must set the charge of the black hole to zero and thus change the geometry from Kerr-Newman to the Kerr geometry. This will not cause much simplification in the Dirac matrices but it gets rid of the vector potential, for which no exact solution exists in the Kerr-Newman geometry, see section 2.3.

We may now calculate the spin connections Γ_μ via (C.11) and thus write down the Dirac equation. The Γ_μ turn out to be very long expressions upon evaluation and we do not stand a chance of separating the Dirac equation in this form. Thus we must resort to a different method.

C.3 Separation and Reduction of the Dirac Equations

"Thus the partial differential equation entered theoretical physics as a handmaid, but has gradually become the mistress."

- Albert Einstein.

Instead of explicitly evaluating the gamma matrices and spin-connections, let us work with them symbolically and use their properties to achieve the separation of the Dirac equation into four separate differential equations for the four components of the Dirac spinor. Consider the full Dirac spinor as being composed to two two-component spinors P^A and Q^A representing the particle and anti-particle respectively. We then introduce a new basis for the spinor space ζ_a^A and for the conjugate space $\zeta_{a'}^{A'}$ following the formalism of Newman and Penrose [26] more formally. Everywhere in the spacetime, there exists a null tetrad associated with the spinor space basis.

One defines a Ricci rotation coefficient or spin coefficient $\gamma_{(a)(b)(c)}$ via the following relationship with the null tetrad.

$$\gamma_{(c)(a)(b)} = e^k_{(c)} \nabla_k \left(e_{(a)i}\right) e^i_{(b)} \tag{C.33}$$

If we consider the basis vectors as directional derivatives, as done so often in general relativity, one may show that the Dirac equation may be reduced to the set of four coupled differential equations [9]

$$\begin{aligned}\left(\partial_{00'} + \frac{1}{2}\left(\gamma_{211} + \gamma_{341}\right) - \gamma_{314}\right) P^0 + \\ \left(\overline{\partial}_{01'} + \gamma_{241} - \frac{1}{2}\left(\gamma_{214} + \gamma_{344}\right)\right) P^1 = i\frac{m}{\sqrt{2}}\overline{Q}^{1'}\end{aligned} \tag{C.34}$$

$$\begin{aligned}\left(\partial_{11'} + \gamma_{243} - \frac{1}{2}\left(\gamma_{212} + \gamma_{342}\right)\right) P^1 + \\ \left(\partial_{01'} + \frac{1}{2}\left(\gamma_{213} + \gamma_{343}\right) - \gamma_{312}\right) P^0 = -i\frac{m}{\sqrt{2}}\overline{Q}^{0'}\end{aligned} \tag{C.35}$$

$$\begin{aligned}\left(\partial_{00'} + \frac{1}{2}\left(\overline{\gamma}_{211} + \overline{\gamma}_{341}\right) - \overline{\gamma}_{314}\right) \overline{Q}^{0'} - \\ \left(\overline{\partial}_{01'} + \overline{\gamma}_{241} - \frac{1}{2}\left(\overline{\gamma}_{214} + \overline{\gamma}_{344}\right)\right) \overline{Q}^{1'} = -i\frac{m}{\sqrt{2}}P^1\end{aligned} \tag{C.36}$$

$$\begin{aligned}\left(\partial_{00'} + \frac{1}{2}\left(\overline{\gamma}_{211} + \overline{\gamma}_{341}\right) - \overline{\gamma}_{314}\right) \overline{Q}^{1'} + \\ \left(\overline{\partial}_{01'} + \overline{\gamma}_{241} - \frac{1}{2}\left(\overline{\gamma}_{214} + \overline{\gamma}_{344}\right)\right) \overline{Q}^{0'} = i\frac{m}{\sqrt{2}}\overline{Q}^{1'}\end{aligned} \tag{C.37}$$

Using the null tetrad for the Kerr geometry (C.25) and the spin coefficients tabulated in [35], it can be shown that these equations are separable when a time dependance of $\exp i\,(\sigma t + m\phi)$ is assumed for all components of the full Dirac spinor. We thus have the solution

$$\Psi = e^{i(\sigma t + m\phi)} \begin{pmatrix} \dfrac{S_{-\frac{1}{2}} R_{-\frac{1}{2}}}{\sqrt{2}\,(r - ia\cos\theta)} \\ S_{\frac{1}{2}} R_{\frac{1}{2}} \\ \dfrac{-S_{\frac{1}{2}} R_{-\frac{1}{2}}}{\sqrt{2}\,(r + ia\cos\theta)} \\ S_{-\frac{1}{2}} R_{\frac{1}{2}} \end{pmatrix} \tag{C.38}$$

where the functions $R_{-\frac{1}{2}}$ and $S_{-\frac{1}{2}}$ satisfy the second order linear ordinary differential equations

$$\partial_\theta^2 S_{-\frac{1}{2}} + Q\,(\theta)\,\partial_\theta S_{-\frac{1}{2}} + R\,(\theta)\,S_{-\frac{1}{2}} = 0 \tag{C.39}$$

and

$$\partial_r^2 R_{-\frac{1}{2}} + F\,(r)\,\partial_r R_{-\frac{1}{2}} + P\,(\theta)\,R_{-\frac{1}{2}} = 0 \tag{C.40}$$

with the coefficient functions given by

$$Q\,(\theta) = \tan\theta + \frac{1}{2}\cot\theta - a\sigma\sin\theta + m\csc\theta \tag{C.41}$$

$$R\,(\theta) = a^2 m^2 \cos^2\theta + \frac{1}{2} - \frac{a\sigma + m}{\cos\theta} + \frac{m\cos\theta - \frac{1}{2}}{\sin^2\theta} \tag{C.42}$$

$$F(r) = i\left[\frac{K}{\Delta} - \frac{m}{\lambda + imr}\right] \tag{C.43}$$

$$P(r) = \frac{mK - (\lambda - imr)\,(\lambda + imr)^2}{\Delta\,(\lambda + imr)} - \frac{2i\,(r - M)\,K}{\Delta^2} \tag{C.44}$$

with

$$K = \left(r^2 + a^2\right)\sigma + am \tag{C.45}$$

The relationship between the two radial functions $R_{\pm\frac{1}{2}}$ may be shown to be

$$\left(\partial_r + \frac{iK}{\Delta}\right) R_{-\frac{1}{2}} = (\lambda + imr)\,R_{\frac{1}{2}} \tag{C.46}$$

Thus we may solve the Dirac equation in Kerr space completely if solutions to the two linear ODE's (C.39) and (C.40) may be found. A complete analytic solution in terms of known functions is not possible but a numerical solution is of course much easier after this separation than it would have been to solve the full coupled partial differential equations that we were faced with initially.

C.4 Assymptotic Solutions

"That queer quantity 'infinity' is the very mischief, and no rational physicist should have anything to do with it. Perhaps that is why mathematicians represent it by a sign like a love-knot."

- Sir Arthur Stanley Eddington.

The solutions to the angular equation do not concern us here, so we will simply assume that it is possible to find a suitably normalised set of orthonormal mode solutions to it. So consider the radial equation in the limit $r \to \infty$. Instead of solving it directly by force consider defining two new functions $\phi_{\pm\frac{1}{2}}$ by

$$\Delta^{\frac{1}{2}} R_{\frac{1}{2}} = \phi_{\frac{1}{2}} \exp\left(-\frac{i}{2}\tan^{-1}\left(\frac{mr}{\lambda}\right)\right) \tag{C.47}$$

and

$$R_{-\frac{1}{2}} = \phi_{-\frac{1}{2}} \exp\left(\frac{i}{2}\tan^{-1}\left(\frac{mr}{\lambda}\right)\right) \tag{C.48}$$

which we can use to construct a wave equation in the new tortoise coordinate

$$\overline{\rho}(r) = \rho(r) + \frac{1}{2\sigma}\tan^{-1}\left(\frac{mr}{\lambda}\right) \tag{C.49}$$

where $\rho(r)$ is the usual tortoise coordinate in Kerr space (B.30). The new wave equation thus sums up the radial solutions to the Dirac equation

$$\left(\frac{d^2}{d\overline{\rho}^2} + \sigma^2\right) Z_\pm = V_\pm Z_\pm \tag{C.50}$$

where $Z_\pm = \phi_{\frac{1}{2}} \pm \phi_{-\frac{1}{2}}$ and the potential is given by

$$\begin{aligned} V_\pm = & \frac{\Delta^{\frac{1}{2}} U^{\frac{3}{2}} \left[\Delta^{\frac{1}{2}} U^{\frac{3}{2}} \pm \left((r-M)U + 3m^2 r\Delta\right)\right]}{\left[\left(r^2 + a^2 + \frac{am}{\sigma}\right)U + \frac{\lambda m \Delta}{2\sigma}\right]^2} \\ & \mp \frac{\Delta^{\frac{3}{2}} U^{\frac{5}{2}} \left[2rU + 2m^2 r\left(r^2 + a^2 + \frac{am}{\sigma}\right) + \frac{\lambda m(r-M)}{\sigma}\right]}{\left[\left(r^2 + a^2 + \frac{am}{\sigma}\right)U + \frac{\lambda m \Delta}{2\sigma}\right]^3} \end{aligned} \tag{C.51}$$

where $U = \lambda^2 + m^2 r^2$. It may be shown that the potential satisfies

$$\lim_{r\to\infty} V_\pm = m^2 \tag{C.52}$$

This enables to find two complete sets of solutions for $\phi_{\frac{1}{2}}$ and thus the radial functions

$$\left\{\phi_{\frac{1}{2}}\right\} = A\exp\left(i\beta\overline{\rho}\right) \qquad \left\{\overline{\phi_{\frac{1}{2}}}\right\} = B\exp\left(-i\beta\overline{\rho}\right) \tag{C.53}$$

where $\beta^2 = \sigma^2 - m^2$. Note the significant difference in the constant β to the solution of similar form to (B.34) for the scalar field in section 2.3.

C.5 The Emission Spectrum

"Although Bekenstein's hypothesis that black holes have a finite entropy requires for its consistency that black holes should radiate thermally, at first it seems a complete miracle that the detailed quantum-mechanical calculations of particle creation should give rise to emission with a thermal spectrum. The explanation is that the emitted particles tunnel out of the black hole from a region of which an external observer has no knowledge other than its mass, angular momentum and electric charge. This means that all combinations or configurations of emitted particles that have the same energy, angular momentum and electric charge are equally probable. Indeed, it is possible that the black hole could emit a television set or the works of Proust in 10 leather-bound volumes ..."

- Stephen W. Hawking.

Making use of the earlier null coordinate connections, we may thus immediately deduce that the emission spectrum of the Dirac field has a temperature

$$T\left(\omega\right) = \frac{1}{4\pi M}\left[\sqrt{1-\left(\frac{m}{\omega}\right)^2} + \frac{M}{r_+ - M}\right]^{-1} \tag{C.54}$$

We observe an essential deviation from the scalar case which takes the form of the minus sign in front of the square of the electron mass. We recall from special relativity (valid in the limit $r \to \infty$ which we have taken) that $E^2 = p^2 + m^2$ and thus $\omega^2 > m^2$, so it is impossible for the temperature to take complex values in spite of the minus sign. Furthermore, we have gained the emission spectrum of the neutrino field for free since it is equivalent to a massless Dirac field with less helicity states. Since these do not matter here, we may simply deduce that the emission spectrum for a massless neutrino is

$$T = \frac{1}{4\pi M}\left[1 - \frac{M}{r_+}\right] \tag{C.55}$$

Taking into consideration the recent results from neutrino observations [11], we should probably not make this distinction and say that (C.54) holds for the neutrino also in which the mass is not quite clear yet but should be around 0.4 electronvolts.

Since the Dirac equation in this form holds for any spin 1/2 particle, we have derived this not only for an electron or a neutrino but any particle with spin 1/2. We see that the temperature is greatest for the particle with $m = 0$ due to the minus sign. Thus the emission probability is highest for massless fermions, while the opposite is true of scalars.

C.6 The Electromagnetic Field

"It does not follow that because something can be counted it therefore should be counted."

- Harold L. Enarson.

The electromagnetic field is described by Maxwell's equations which, in general relativity, are recast into a tensor formalism in which the object of interest is the Faraday tensor $\mathcal{F}_{ab}$ which is a combination of the components of the electric field E and the magnetic field B, such that

$$\mathcal{F}_{ab} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{C.56}$$

and Maxwell's equation may be shown to reduce to two tensor equations in $\mathcal{F}$

$$\nabla_a \mathcal{F}_{bc} + \nabla_b \mathcal{F}_{ca} + \nabla_c \mathcal{F}_{ab} = 0; \qquad a \neq b \neq c \tag{C.57}$$

and

$$\nabla_b \mathcal{F}^{ab} = j^a \tag{C.58}$$

where $j_a = (\rho, \mathbf{j})$ is the four-current. In curved spacetime, the covariant derivatives are complicated differential operators like the d'Alembertian (A.39) and thus these equations are rather complex. We must separate all these equations and then solve them for the field components in order to be able to derive the emission spectrum. Fortunately, Chandrasekhar has done this for a zero external electric field and we just need to take his solution that assymptotically ($r \to \infty$) the electric and magnetic fields vary with a characteristic scalar which varies just as the Dirac field varies. The derivation of this result is very complex and the reader is referred to Chandrasekhar [8] for a discussion of it. Since the solution is for a zero external field, we must work with the connection of the null coordinates for Kerr spacetime and thus the result is exactly the same as for the massless Dirac field, i.e. the neutrino. Thus we find that the emission temperature for a photon is

$$T = \frac{1}{4\pi M}\left[1 - \frac{M}{r_+}\right] \tag{C.59}$$

Appendix D

The Binary System

"Space is the essential basis for the appearance of physical structures."

- Tarthang Tulku.

From observations, we know that often black holes are not far enough apart from other stars so that we may neglect the influence of the stars upon the black holes. Some of the best candidates for black holes are members of a binary system in which the partner is certainly important. We wish to study how the inclusion of several black holes might change the scalar emission spectrum. We shall find that even though we will treat the situation for two black holes, the derivation is general for any number of black holes, provided that they form a finite cluster and the observer is far away compared to the distances between the black holes.

D.1 The Metric

"There is nothing in the world except curved empty space. Geometry bent one way here describes gravitation. Rippled another way somewhere else it manifests all the qualities of an electromagnetic wave. Excited at still another place, the magic material that is space shows itself as a particle. There is nothing that is foreign and 'physical' immersed in space. Everything that is, is constructed out of geometry."

- John A. Wheeler.

An explicit solution for the metric of two black holes may be found [20]

$$dl^2 = -f^{-1}\gamma_{mn}dx^m dx^n + f\left(\omega_m dx^m + dt\right)^2 \tag{D.1}$$

where

$$f = \left|1 + \sum_{i=1}^{N} \frac{M_i}{R_i}\right|^2 \tag{D.2}$$

and

$$\gamma_{mn} = \mathrm{diag}\left[1, r^2, r^2 \sin^2\theta\right] \tag{D.3}$$

This result holds for N black holes with masses M_i. When $N = 2$ and the positions of the black holes are $(0, 0, b_1)$ and $(0, 0, -b_2)$ with $b_1, b_2 > 0$, then

$$R_j^2 = r^2 - 2\left(b_j + ia_j\right) r\cos\theta + \left(b_j + ia_j\right)^2 \tag{D.4}$$

It may be shown [20], that we may require the following properties of the ω's

$$\omega_r = \omega_\theta = 0; \qquad \lim_{r\to\infty} \omega_\phi = 0 \tag{D.5}$$

and we may easily see from (D.2) that

$$\lim_{r\to\infty} f = 1 \tag{D.6}$$

The function ω_ϕ may be evaluated explicitly [20] but we shall only need to make use of its assymptotic limit.

We then find the metric tensor, its determinant and its inverse from the (D.1) - (D.4) to be

$$g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 & f\omega_\phi \\ 0 & -f^{-1} & 0 & 0 \\ 0 & 0 & -\dfrac{r^2}{f} & 0 \\ f\omega_\phi & 0 & 0 & f\omega_\phi^2 - \dfrac{r^2\sin^2\theta}{f} \end{pmatrix} \tag{D.7}$$

$$g \equiv \det\left(g_{\mu\nu}\right) = -r^4 f^{-2} \sin^2\theta \tag{D.8}$$

$$g_{\mu\nu} = \begin{pmatrix} f^{-1} - \dfrac{f\omega_\phi^2}{r^2\sin^2\theta} & 0 & 0 & \dfrac{f\omega_\phi^2}{r^2\sin^2\theta} - f^{-1} \\ 0 & -f & 0 & 0 \\ 0 & 0 & -\dfrac{f}{r^2} & 0 \\ \dfrac{f\omega_\phi^2}{r^2\sin^2\theta} - f^{-1} & 0 & 0 & -\dfrac{f}{r^2\sin^2\theta} \end{pmatrix} \tag{D.9}$$

Using (D.7) - (D.9) we may use (A.39) to find the d'Alembertian operator in this spacetime

$$\begin{aligned}\Box^2\Phi = &\left(f^{-1} - \frac{f\omega_\phi^2}{r^2\sin^2\theta}\right)\partial_t^2\Phi + 2\left(\frac{f\omega_\phi^2}{r^2\sin^2\theta} - f^{-1}\right)\partial_t\partial_\phi\Phi \\ &- \frac{f}{r^2\sin^2\theta}\partial_\phi^2\Phi - \frac{f}{r^2}\partial_r\left(r^2\partial_r\Phi\right) - \frac{f}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right)\end{aligned} \tag{D.10}$$

D.2 The Scalar Field

"It is the theory that decides what we can observe."

- Albert Einstein.

The Klein-Gordon equation (A.38) where $\Box^2$ is given by (D.10) is satisfied by the wavefunction

$$\Phi = \exp\left(-i\omega t\right) Y_{ln}(\theta,\phi)R(r) \tag{D.11}$$

where $Y_{ln}(\theta,\phi)$ are the spherical harmonics and the radial wavefunction $R(r)$ satisfies the equation

$$\begin{aligned}&\partial_r^2 R + \frac{2}{f}\partial_r R - \left(2\omega n + \omega^2\right)\left(\frac{\omega_\phi^2}{r^2\sin^2\theta} - f^{-2}\right)R \\ &\qquad + \left[\frac{n^2}{r^2\sin^2\theta} + \frac{m^2}{f} - l(l+1)\right]R = 0\end{aligned} \tag{D.12}$$

If we transfer to a tortoise coordinate

$$\rho(r) = \frac{r^2 + \sum_i a_i^2}{r^2 - \sum_i\left(2M_i a_i - a_i^2 - Q_i^2\right)} \tag{D.13}$$

we may simplify (D.12) in the limit as $r \to \infty$ to the equation

$$\partial_\rho^2 R\left(\rho\right) + \alpha^2 R\left(\rho\right) = 0 \tag{D.14}$$

where $\alpha^2 = 2\omega n + \omega^2 + m^2 - l(l+1)$. If we take our null coordinates to be

$$u = t - \frac{\alpha}{\omega}\rho \qquad v = t + \frac{\alpha}{\omega}\rho \tag{D.15}$$

we have the two solutions as before in terms of the two null vectors

$$u_i = u_{\omega ln}^{(in)} = e^{-i\omega v} Y_{ln}(\theta, \phi) \quad \text{(D.16)}$$

$$\overline{u}_i = u_{\omega ln}^{(out)} = e^{-i\omega u} Y_{ln}(\theta, \phi) \quad \text{(D.17)}$$

It may be shown, analogously to the derivation in section 2.1 for the single Schwarzschild black hole , that these solutions are orthogonal and can be properly normalised with repect to the conditions (A.29). For this, it is vital to remember that the scalar product is independant of the choice of hypersurface.

D.3 The Emission Spectrum

"According to 'Turner's Law,' the invocation of the tooth fairy should not occur more than once in any scientific argument."

- Marcia Bartusiak.

When dealing with one black hole , we obtained scalar modes of the same form as the present modes. It was found that if we could find one null coordinate in terms of the other one, we could find the Bogolubov coefficients without actually having to do a very complicated integral. We were able to use a different contour in the complex plane to transform one integral into a multiple of the other and then, via the normalisation condition, find the Bogolubov coefficient. If we want to apply the same method here, we must find $u = u(v)$. Two isolated black holes can not be treated as two separate collapsing shells of matter since then, the emission spectrum would depend on which exact path the null coordinates took; that is if they intersected both or only one black hole. There is no simple local way of determining the connection. Since we are only interested what an observer at infinity sees, we may treat the cluster accordingly. When the observer is far away from the black hole cluster, a number of black holes act as one with the sum of the properties (Gauss' Theorem) [21]

$$M = \sum_i M_i \qquad a = \frac{1}{M} \sum_i |\mathbf{J}_i| \qquad Q = \sum_i Q_i \quad \text{(D.18)}$$

Thus we may in fact use the same connection procedure as before with the new parameters given by (D.18). We then have

$$u(v) = -4M\gamma \ln \left[\frac{v_0 - v}{Cr_+} \right] \quad \text{(D.19)}$$

where

$$\gamma = \frac{1}{2}\left[\frac{\alpha}{\omega} + \frac{M}{r_+ - M}\right] \tag{D.20}$$

We now employ the same method as in section 2.2 to find the temperature of the emission spectrum. Since the form of (D.19) is identical to the form of (B.37), we may use the result for the temperature from (B.39) and thus have

$$T(\omega) = \frac{1}{4\pi M}\left[\frac{\alpha}{\omega} + \frac{M}{\sqrt{M^2 - a^2 - Q^2}}\right]^{-1} \tag{D.21}$$

in terms of the parameters given by (D.18). Note that the emission spectrum depends upon the angular quantum numbers l and n of the field as well as the angular frequency ω and the mass of the scalar m. The emission temperature is the same for any number N of black holes, such that N ¿ 1. That it depends upon the angular quantum numbers is a new effect and will lead to structure in the emission spectrum. To find the total emited number of scalars with a particular frequency, we must sum over the modes that have this frequency.

$$N_\omega^{tot} = \sum_{l=0}^{\infty}\sum_{n=-l}^{l}\left[exp\left(\frac{\omega}{T}\right) - 1\right]^{-1} \tag{D.22}$$

The sum over n is limited by the properties of the spherical harmonics but the sum over l is not limited as it would be in the hydrogen atom, for example. This sum must be evaluated numerically.

Appendix E

Interacting Fields

"Physics doesn't have to have any use. It just is."

- Robert A. Heinlein.

So far we have only considered free fields. Measurements however, can only be made if the field interacts with either itself or another field because if no interactions are present, for all practical purposes, the field does not exist. We have been able to make several very interesting predictions based upon the free field treatment and we expect this to be the major contribution to the particle production. It is interesting to study what the effect of interactions would be upon the field emission spectrum from a black hole. In the following sections we shall consider the interaction of the scalar field with the gravitational field and itself.

E.1 The Gravitational Interaction

"If A is success in life, then A equals x plus y plus z.
Work is x; y is play and z is keeping your mouth shut."

- Albert Einstein.

The interaction of any field with the gravitational one is treated through the introduction of an extra term in the Lagrangian density for the field

$$\mathcal{L}_{int} = -\frac{1}{2}\xi R \Phi^2 \tag{E.1}$$

where R is the Ricci scalar defined in terms of the Riemann tensor (A.11)

$$R = g^{\alpha\beta} R^{\gamma}_{\beta\gamma\alpha} \tag{E.2}$$

Upon invoking the action principle, we would get an extra term $\xi R \Phi$ in the field equations. In the case of the Klein-Gordon equation, we could thus regard the gravitational field interaction as introducting a position dependant mass of the field

$$(m')^2 = m^2 + \xi R \tag{E.3}$$

From experience, two values of ξ are particularly important. The minimally coupled case, $\xi = \xi_m = 0$, and the conformally coupled case,

$$\xi = \xi_c = \frac{n-2}{4(n-1)} \tag{E.4}$$

where n is the number of dimensions of the spacetime [13]. In the case of normal general relativity, we have $n = 4$ and so $\xi_c = 1/6$. Many models of black holes are done in two dimensions since the separation of the field equations is so much easier, in this case $\xi_c = \xi_m = 0$. The conformally coupled case is named so because if $m = 0$ and ξ is given by (E.4), then the field equation will be invariant under conformal transformations of the metric, i.e.

$$g_{\mu\nu}(x^{\mu}) \longrightarrow \Omega^2(x^{\mu})\, g_{\mu\nu}(x^{\mu}) \tag{E.5}$$

By (A.12), we may determine that for any black hole spacetime

$$\lim_{r \to \infty} R = 0 \tag{E.6}$$

This property allows us to deduce that when any wave-equation is derived from its Lagrangian combined with the interaction Lagrangian (E.1), we will get an assympotic equation which does not depend upon ξ. Thus, since we are only concerned with the region in which the radial coordinate tends to infinity, the gravitational interaction has no effect upon the emission of *any* type of particle. Taking the limit to infinity is not strictly permissable since an observer is never an infinite distance away from anything observable. Thus we should of course do the entire solution for a finite radial coordinate r. The complication is that none of the differential equations would then be soluble and thus the endeavour would be a numerical one. If the system is sufficiently well behaved, we may postulate that the result derived for $r \to \infty$ holds to a good order of approximation when the observer is at a distance $r \gg r_E$, the event horizon of the black hole.

E.2 The Self Interaction

"A theory, to be of any real use to us, must satisfy two tests. In the first place, it must not make use of any ideas which are not confirmed by experiment. Special assumptions must not be dragged in merely to meet some particular difficulty. In the second place, the theory must not only explain all the facts known already, but must also enable us to forsee other facts which were not known before and can be tested by further experiment."

- Max Born.

Like the gravitational interaction , the interaction of a field with itself is treated through an new term in the lagrangian density

$$\mathcal{L}_{self} = -\frac{1}{4}\lambda\Phi^4 \tag{E.7}$$

Upon applying the action principle, this Lagrangian introduces the term $\lambda\Phi^3$ into the field equations. This nonlinearity will complicate matters greatly since we must now solve a nonlinear partial differential equation instead of a linear one. This means that in general the variables may no longer be separated and we can not adopt the solution methods of the previous chapters. When the coupling constant λ is small, we may be able to find a solution using asymptotic methods and then discuss the solution in the phase plane of the differential equation. This will not help us very much in the endeavour to calculate the Bogolubov coefficients, however.

What we may be able to do is to introduce another variable into the partial differential equation to transform it into an ordinary differential equation (ODE) which may be soluble. The resultant ODE will, of course, still be nonlinear to the third degree but at least we may adopt the usual methods of solving ODE's to search for a solution. Because of the complexity, we shall study only the Klein-Gordon field with this interaction.

E.3 The Scalar Field

"Mathematics alone satisfies the mind through absolute certainty."

- Johnnes Kepler.

The Klein-Gordon equation in the presence of non-zero self interaction becomes

$$\left(\Box^2 + m^2\right)\Phi + \lambda\Phi^3 = 0 \tag{E.8}$$

Since the variables in this equation can not be separated, we immediately transform to the tortoise coordinate ρ given by (B.30) and take the limit $r \to \infty$ to obtain the equation

$$\left(\partial_\rho^2 - \partial_t^2 + m^2\right)\Phi + \lambda\Phi^3 = 0 \tag{E.9}$$

We may transform this into an ODE by transforming to the variable $s = t \pm \mu\rho$ to get

$$\left(\mu^2 - 1\right)\frac{d^2\Phi}{ds^2} + m^2\Phi + \lambda\Phi^3 = 0 \tag{E.10}$$

which may be put into a more standard form

$$\frac{d^2\Phi}{ds^2} + w^2\Phi + \beta\Phi^3 = 0 \tag{E.11}$$

The ODE (E.11) possess soliton solutions ([29] and [18]) which may however be shown not to be orthogonal or normalisable according to the conditions (A.29). These soliton solutions may be of use when dealing with the self-interacting scalar field under different conditions. For instance if a solution describing one particular particle were sought, these soliton solutions would be perfect. For our purposes, we must search for a complete solution of the equation and extract solutions which are properly normalisable and orthogonal. A first integral of (E.11) may be found by standard methods

$$\left(\Phi'\right)^2 + w^2\Phi^2 + \frac{\beta}{2}\Phi^4 = C_1 \tag{E.12}$$

where C_1 is a constant. This first order ODE may be discussed in the phase plane [34]. The phase plane analysis shows many properties of the solutions and would be a good tool to find characteristics of the field in different circumstances. To find the emission spectrum, we need an explicit solution, however. Rearranging, the solution may be expressed as an integral

$$s + C_2 = \pm\sqrt{\frac{-2}{\beta}}\int^{\Phi}\left[\Phi^4 + \frac{2w^2}{\beta}\Phi^2 - \frac{2C_1}{\beta}\right]^{-\frac{1}{2}} d\Phi \tag{E.13}$$

E.4 Elliptic Solutions

"Your calculations are correct, but your physics is abominable."

- Albert Einstein.

To evaluate (E.13), we put $C_1 = \frac{w^4}{2\beta}\left(D^2 - 1\right)$ and use the properties of the Jacobi elliptic functions (for a tabulation see [1] and for a discussion see [6]), we may find that

$$s + C_2 = \pm\sqrt{\frac{-2}{\beta}}\int d\Phi\left[\left(\Phi^2 - \frac{w^2}{\beta}(D-1)\right)\left(\Phi^2 + \frac{w^2}{\beta}(D+1)\right)\right]^{-\frac{1}{2}} \quad \text{(E.14)}$$

$$= \pm\frac{i}{w\sqrt{D}}\mathrm{cn}^{-1}\left[\frac{w\sqrt{D-1}}{\sqrt{\beta}\Phi},\quad \sqrt{\frac{D+1}{2D}}\right] \quad \text{(E.15)}$$

Inverting this solution, we obtain the complete solution for the field in terms of the two arbitrary constants C_2 and D.

$$\Phi = w\sqrt{\frac{D-1}{\beta}}\mathrm{cn}\left[\pm w\sqrt{D}\left(s + C_2\right),\quad \sqrt{\frac{D-1}{2D}}\right] \quad \text{(E.16)}$$

To put this into a more convinient form, put $F = \sqrt{\frac{D-1}{2D}}$. Then we must find two sets of solutions for Φ such that they obey the orthonormality conditions (A.29). Directly this will not be possible since the solution (E.16) is a real function. Even if we are able to find two complete sets of modes, they will not satisfy the original conditions since they will be real. To remedy this difficulty, we must reconsider the first assumptions in being able to expand one set of solutions in terms of the other. We had

$$u_i = \sum_j\left(\alpha^*_{ji}\bar{u}_j - \beta_{ji}\bar{u}^*_j\right) \quad \text{(E.17)}$$

but since the solutions are real, the Bogolubov coefficients must be real and we now have

$$u_i = \sum_j\left(\alpha_{ji} - \beta_{ji}\right)\bar{u}_j \quad \text{(E.18)}$$

$$= \sum_j\varepsilon_{ji}\bar{u}_j \quad \text{(E.19)}$$

We may now find that the emission spectrum is given by

$$N_i = \sum_j\left|\beta_{ji}\right|^2 = \frac{1}{4}\sum_j\varepsilon^2_{ji} \quad \text{(E.20)}$$

Thus we need not throw away our solution (E.16) but we may determine the emission spectrum in spite of the fact that the solution is real. This also implies that our field solution only needs to satisfy one orthonormality condition,

$$(u_i, u_j) = \delta_{ij} \tag{E.21}$$

instead of three. Using the properties of the Jacobi elliptic functions, it can be shown that for the solutions to be properly normalised the constant F must be chosen such that

$$\frac{-4w^2}{\beta\left(1-2F^2\right)}\left[2E\left(\frac{\pi}{2},F\right)-2\left(1-F^2\right)K\left(\frac{\pi}{2},F\right)\right]=1 \tag{E.22}$$

where $E\left(\frac{\pi}{2},F\right)$and $K\left(\frac{\pi}{2},F\right)$ are the Jacobi elliptic integrals of the first and second kind defined by

$$E\left(\frac{\pi}{2},F\right)=\int\limits_0^{\pi/2}\frac{d\theta}{\sqrt{1-F^2\sin^2\theta}} \tag{E.23}$$

$$K\left(\frac{\pi}{2},F\right)=\int\limits_0^{\pi/2}\sqrt{1-F^2\sin^2\theta}d\theta \tag{E.24}$$

The Jacobi elliptic function $\mathrm{cn}(x,F)$ behaves as the cosine function with the period $K = E\left(\frac{\pi}{2},F\right)$. Using this property, we may infer that $\mathrm{cn}(x,F)$ is orthogonal just like the cosine function. This fact may in fact be demonstrated numerically and thus we have

$$\int\limits_{-2K}^{2K}\Phi_w\Phi_{w'}ds=\delta_{ww'} \tag{E.25}$$

for the choice $C_2 = 0$ and D being fixed by the normalisation requirement (E.22). In order for the solutions to be fully orthonormal like the cosine function, we must require, in addition to (E.22), that the coefficient of the independant variable $\omega\sqrt{D} = \omega/\sqrt{1-2F^2}$ is an integer. We have chosen F already and thus we must impose the condition that ω can only take certain values given by

$$\omega = p\sqrt{1-2F^2} \tag{E.26}$$

where p is a positive integer or zero. We note that since the variable s has two possibilities built into it, we already have two complete sets of solutions to the nonlinear Klein-Gordon field which are suitably normalised. We may thus identify the two null coordinates as the two possibilities for s and put

$$u = t - \left(\frac{m^2}{\omega^2} + 1\right)^{\frac{1}{2}} \rho, \qquad v = t + \left(\frac{m^2}{\omega^2} + 1\right)^{\frac{1}{2}} \rho \tag{E.27}$$

where we have chosen the separation constant $\mu = \sqrt{m^2/\omega^2 + 1}$ as we had before. The null coordinate u may be found as a function of v by the method discussed in section 2.4. Thus we may evaluate the single Bogolubov coefficient

$$\varepsilon_{\omega\omega'} = \int_{-2K}^{2K} \Phi_\omega \overline{\Phi}_{\omega'} d\rho \tag{E.28}$$

and the emission spectrum is then

$$N_\omega = \sum_{\omega'} \varepsilon_{\omega\omega'} d\omega' \tag{E.29}$$

Both (E.28) and (E.29) must be evaluated numerically for a specific black hole. Without evaluating either explicitly, we may already argue that the spectrum will be significantly altered since the solution (E.16) will not reduce to our previous solution upon letting the self-coupling constant $\lambda \to 0$. This is a currious property of the solution but one which occurs frequently in the solutions of differential equations. On physical grounds, we must expect a small deviation from the previous emission spectrum but on mathematical grounds, there is no reason for this to be so. The main difference is, of course, the "quantisation" of the field frequency.

Appendix F

Physical Discussion

"Physics does not explain the secrects of nature, it leads to deeper secrets."

- Carl Friedrich von Weizäcker.

In the preceeding chapters, we have derived the emission spectra of a variety of particles by black holes. Amongst all this mathematical derivation of the results we have not discussed the meaning of the results obtained in detail and have not given the emission spectra in normal SI units. In this section 1, we shall give all of this information and compare and contrast the main features of the emission spectra for a various types of particles. Section 2 will discuss the famous information paradox which concerns the possible conservation (or otherwise) of information in the light of the results gained in this paper. Section 3 sets the paper in context of present research and suggests several ways in which the work began here could be carried on to produce meaningful new physics.

F.1 Comparison of Spectra

"Indeed, modern theoretical physics is constantly filling the vacuum with so many contraptions that it is amazing a person can even see the stars on a clear night"

- M.J.G. Veltman.

All the mathematics in this paper so far has been done in geometrised units in which the speed of light in the vacuum c, the gravitational constant G, Boltzman's constant k, the permittivity of the vacuum ϵ_0 and the permeability of the vacuum μ_0 are all equal to one. Furthermore, we represented the mass of the particle by the inverse of its Compton wavelength, so $m = \hbar m_0/c$ where m_0 is the rest mass of the particle. The values of the constants used below are given in table 6.1. The transition from geometricised units into SI may be accomplished via the transformations listed in table 6.2.

Table 6.1: SI Values of Constants

Constant	Value	Units
c	$2.9979 \cdot 10^{10}$	cm s^{-1}
G	$6.6700 \cdot 10^{-8}$	cm^2 g^{-1} s^{-2}
k	$1.3805 \cdot 10^{-23}$	J K^{-1}
$\hbar$	$1.0545 \cdot 10^{-34}$	J s
q	$1.6020 \cdot 10^{-19}$	J eV^{-1}
ε_0	$8.8542 \cdot 10^{-12}$	C V^{-1} m^{-1}
$\frac{\hbar c^3}{4\pi k G}$	$2.4553 \cdot 10^{26}$	g K
$q/\hbar$	$1.5194 \cdot 10^{15}$	eV^{-1} s^{-1}
G/ε_0	$7.5331 \cdot 10^{12}$	cm^6 s^{-4} C^{-2}
$\hbar/k$	$7.6378 \cdot 10^{-12}$	s K

Table 6.2: Transformation of Units

Geometric Variable	SI Variable
M	GM/c^2
a	a/c
Q	$\sqrt{G/\varepsilon_0 c^4}Q$
m	$m/qc\hbar$
ω	ω/c

Making use of these transformation properties of the variables and parameters of the emission spectra, we may express the number of particles emitted into the mode of angular frequency ω by

$$N_\omega = \left[\exp\left(\frac{\hbar\omega}{kT}\right) - 1\right]^{-1} \tag{F.1}$$

where in general $T = T(\omega)$. The exact temperatures for all the various fields can then be collected together into two formulae

$$T(\omega) = \frac{\hbar c^3}{4\pi k G M}\left[\sqrt{1+\frac{2nc}{\omega}-\frac{c^2 l(l+1)}{\omega^2}+\left(\frac{qm}{\hbar\omega}\right)^2}+\frac{GM}{\sqrt{G^2M^2-GQ^2/\epsilon_0-c^2a^2}}\right]^{-1} \tag{F.2}$$

and

$$T(\omega) = \frac{\hbar c^3}{4\pi k G M}\left[\sqrt{1-\left(\frac{qm}{\hbar\omega}\right)^2}+\frac{GM}{\sqrt{G^2M^2-c^2a^2}}\right]^{-1} \tag{F.3}$$

where the units of the variables are

$$[M]=g,\ [a]=cm^2s^{-1},\ [Q]=C,\ [m]=eV,\ [\omega]=Hz \tag{F.4}$$

Table 6.3 shows the particles to which these formulae apply and any special settings of the parameters required.

Table 6.3: Temperature Identification

Particle	Geometry	Temperature	Parameter Requirements
scalar	Schwarzschild	(F.2)	$a = Q = n = l = 0$
neutral scalar	Kerr-Newman	(F.2)	$n = l = 0$
neutral scalar	Cluster	(F.2)	none
electron	Kerr	(F.3)	none
neutrino	Kerr	(F.3)	$m = 0$
photon	Kerr	(F.3)	$m = 0$

From tables 6.1 to 6.3, we may now calculate the emission temperature in degrees Kelvin for almost all known particles. It is remarkable that one should be able to reduce all the different field solutions for different situations to two very similar equations for the emission temperature.

We note the magnitude of the temperature first of all. For a minimum size black hole of three solar masses, the scalar emission temperature is of the order of 10^{-10} Kelvin. Such a low temperature will cause hardly any emission at all. For an emission temperature equal to room temperature, a black hole of $2 \cdot 10^{-12}$ solar masses is necessary. Such a mini black hole could only have been formed at the time of the big bang and would be evaporating around this time in the universal evolution.

For a single Kerr black hole, the only difference between the two emission temperatures is a change of sign before the square of the particle mass. It is obvious that as the frequency becomes large, the temperatures tend to the same value. At low frequencies, it is the Dirac emission which dominates however.

When the charge and angular momentum of a Kerr-Newman black hole are small, the temperature approaches that of the Schwarzschild solution. When they are large, all that is changed is that the limit of the temperature as the frequency becomes large, decreases. It may thus be deduced that the Schwarzschild solution has the largest emission temperature.

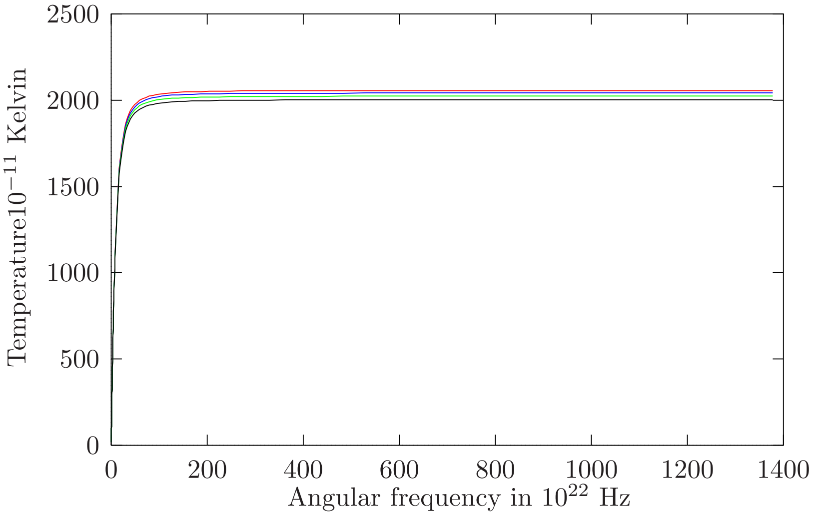

Figure 6.1: The emission temperature for electron emission from a Kerr black hole of three solar masses. The red, blue, green and black curves correspond to values of the angular momentum per unit mass $a = 0$, $0.5M$, $0.75M$ and $0.95M$, in geometrical units, respectively.

For a cluster of black holes, the temperature depends upon the angular quantum numbers l and n which are chosen such that $-l < n < l$. For any reasonable choice of l, we may thus see that the value of n dominates. The limit as the frequency becomes large is the same for all different angular modes. For finite frequencies, the temperature is decreased with increasing quantum numbers. We would expect this since it should be harder to emit a particle in a high state than a low state. It can however be verified numerically, that the temperature is a very weak function of the angular quantum numbers, so that for conservative choices of l and n, we may regard them as neglegible parameters.

Since the temperatures depend on so many parameters, it is difficult to illustrate these points graphically. In one figure 6.1, we compare the temperatures for electron emission from a Kerr black hole for various values of the angular momentum for a black hole at the Chandrasekhar limit (a mass of three solar masses). The red, blue, green and black curves correspond to values of the angular momentum per unit mass $a = 0$, $0.5 \cdot M$, $0.75 \cdot M$ and $0.95 \cdot M$, in geometrical units, respectively. Figure 6.2 shows the emission number for three scalar particles from a mini Schwarzschild black hole of mass $M = 5.97 \cdot 10^{13}$ grams. The red, blue and green curves correspond to the neutral pion, neutral kaon and eta respectively.

We see that there is little variation with changing angular momentum until we get close to the extremal case, where $a = M$. In this limit, the emission temperature tends to zero. We also observe that the emission of lighter particles is prefered. This is intuitive but it is good that this is brought out in the mathematics.

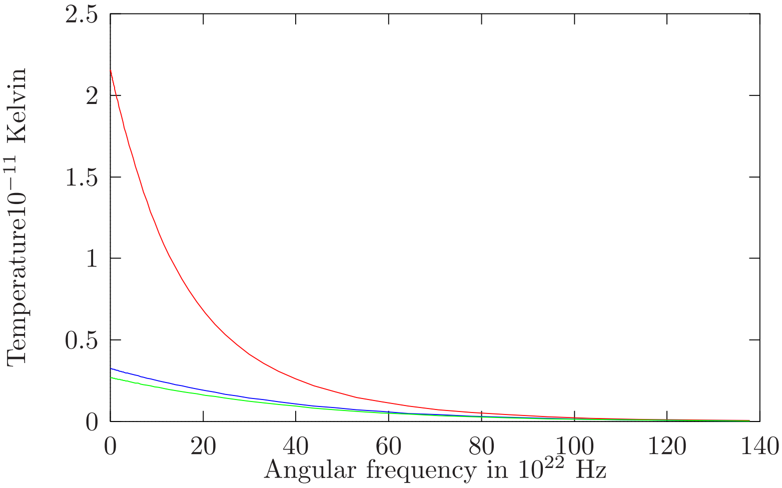

Figure 6.2: Emission spectrum for three scalars from a Schwarzschild black hole of mass $M = 5.97 \cdot 10^{13}$ grams. The red, blue and green curves correspond to the neutral pion, neutral kaon and eta respectively. The masses are $\pi^0 = 135$ MeV, $\kappa^0 = 498$ MeV and $\eta = 549$ MeV.

F.2 Observability of Black Hole Evaporation

The emission of particles from a black hole may be observable if the emission is frequent enough so that many particles would arrive at a telescope on earth. From the general form of the emission law (F.1), we note that this occurs when

$$\hbar\omega \approx kT \tag{F.5}$$

We see that certain modes of emission are thus preferred over others. In such a way, one may, for a certain black hole, determine the 'favorite' emission mode for a certain field.

When the mass of the particle is small compared to the frequency of the field, the temperature is approximately constant and we observe a Planck spectrum. How then might we distinguish a black hole emission spectrum from any other Planck spectrum such as the microwave background radiation? We take the example of the binary neutron star system PSR 1913+16 wherein the slowing down of the rotational motion due to gravitational radiation was measured to a greater accuracy than any result in QED. If it were possible to observe several small black holes and measure their mass, charge and angular momentum with very high accuracy, one may be able to see the reduction of these parameters with time due to particle emission. If one were to look directly at a black hole and measure its spectrum, one would probably not be able to observe a significant signal because of radiation due to other objects.

By looking at the temperatures, we note that the emission of particles becomes stronger the smaller the black hole and that it will not stop until the black hole has fully evaporated. We may not actually use our results to predict the final disappearance of the black hole since we are dealing with the approximation that we may treat gravity classically. This assumption is only true when the curvature, as measured by the Ricci scalar, is small compared to the wavelength of the quantum field. When a black hole becomes arbitrarily small, this assumption no longer holds. Because this is an approximation, we may expect it to be the dominant behaviour of the process and so we can reasonably expect that the black hole will actually finally disappear, even though we can not say for certain until a full quantum theory of gravity is available.

There has been a proposal that after the big bang, when the matter was very dense, mini black hole could have formed with masses down to the Planck mass. Such black holes would now be close to the final evaporation point and radiating vigorously. It has been conjectured that the emission of particles during the final few seconds would be powerful enough to compete with a supernova for luminosity. Such an event would thus certainly be observable.

The likelihood of there being any such mini black holes is not determined but there would need to be a great number of different masses in order for astronomers to have reasonable chances of picking up such explosions. The lack of such observations points towards a dearth of mini black holes created in the big bang. This is the first concrete physical deduction that we can draw from the theory and experiment combined.

Since we did not have to assume that we are dealing with an actual black hole, this effect would also exist for any object with a metric of the Kerr-Newman type, which is any stellar object. Since these are generally massive and sparse, the number of emitted particles will be low and thus the effect is neglegible.

F.3 The Information Paradox

"Every proposition can be viewed from two points of view, a scientific and a sensible one."

- A. Bier.

Every system carries information in it. A single particle has position and momentum, thus six pieces of independant information. Also the particle has a mass, charge, spin and possible other quantum numbers that distinguish it from other particles. All this is the information content of the particle. In a complex system, such as a star, there is much more information. When such a star collapses into a black hole, all this information is lost along with all the matter. However the black hole is described only by its mass, angular momentum and charge in addition to its position and linear momentum. Since the black hole is characterised by so few parameters, so is the metric and thus the emission spectrum for a quantum field. This in turn means that little information leaks out of the black hole. If the black hole ever evaporated completely, where does this information remain?

There are a number of easy answers to this question. First, information is not conserved and thus the question is irrelevant; the information simply disappears. Second, black holes do not completely evaporate and thus the information remains within. Third, the black hole singularity forms a gateway to another part of the universe or to another universe altogether through which the information leaks. Let us then discuss each of these possible answers.

Philosophically, one would like information to be conserved. This is a consequence of the second law of thermodynamics which states that the entropy of a closed system, such as the universe, either stays constant or increases. We may identify the concept of entropy with information and thus deduce that the information content of the universe must either remain constant or increase over time. According to the law of simplicity or Ockham's razor, we then postulate conservation of information which would limit the change of universal entropy to zero. Even a general second law would speak against the first explanation of the paradox since information has apparently sharply decreased.

The second explanation presumes the second law and attempts to uphold it by requiring a full theory of quantum gravity to stop the emission process when a black hole approaches the Planck mass. This would provide us with another achievement for free: An explanation for the cosmic censor. Penrose showed that naked singularities make a spacetime unphysical but also showed that singularities are inevitable [16]. Thus he conjectured that a cosmic censor exists that masks naked singularities. A quantum gravitational plug on the Hawking effect is the perfect mechanism to ensure that such censorship is carried out. This would even explain the current null result on observed black hole explosions.

The third explanation also assumes the second law. Now we enforce it by emitting the information through either a white hole in another universe (for example region II in figure 1.1) or an Einstein-Rosen bridge to another part of this universe. This would allow black hole to fully evaporate and still maintain the second law. The cosmic censor would be unhappy, but we may find another way of preventing that.

It seems that the second explanation is a very good one. It upholds what we know without introducing anything too much like startrek at the expense of a new physical effect from a theory as yet undiscovered. This may be too much of a price to pay. From our derivations, it would seem that black holes would evaporate fully and there is no hint at what may stop this from occuring. Furthermore, our deductions should be the first order calculations to the emission but we have said that this does not hold for Planck sized black holes. It would thus seem perfectly reasonable to require that quantum gravity prevent the emission of particles from small black holes.

F.4 Directions in the Future

"Science really starts to get interesting when it stops."

- Freiherr Justus von Liebig.

Constrained by time, we have not been able to exhaust the topic of Hawking radiation by far. Much could be done and there is cause to attempt to do some of these things in the future. Let me name a few problems not yet solved for motivation of future research.

We have considered only the gravitational interaction for all fields and the self interaction for the scalar field. Interacting fields are much different from free ones and one may deduce far more from them. It would increase our knowledge of quantum fields in curved spacetimes by a great deal if interacting fields could be studied in detail. All fields are simultaneously present around a black hole and that must cause some additional effects.

The effect of the reduction in mass will have an effect upon the metric of the black hole. This back reaction effect becomes important for small black holes and may be another source for new physics. We have made no attempt at including this very difficult topic into our deductions here. Treating this topic in detail would require much time but good results would come of it.

The solutions presented here have relied on assymptotic approximations. When the distance of the observer to the black hole can not be assumed to be infinity, the emission spectrum will no longer be Planck. It is possible that the limit of this new spectrum tends to the Planck one so slowly that the spectrum as seen is in no way like the spectrum we have derived. Unfortunately, the differential equations can not be solved exactly for a finite radial coordinate and thus this solution must be done numerically.

On a more formal level, an investigation into the properties of the Bogolubov coefficients and emission spectrum based upon general properties of the metric tensor would be very revealing. A general investigation of particle emission by curvature would reveal that this effect is not limited to black holes but merely easier to observe because the curvature is greater near the event horizon than anywhere else.

Research into quantum fiel theory in curved spacetimes is necessary to obtain a more complete picture of fundamental influences of gravity upon quantum events. It also allows us to approximate quantum gravity without ever having it. Having concrete and verifiable physical predictions from QFTCS will make the hunt for quantum gravity easier since it must reduce to it in the small curvature limit.

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